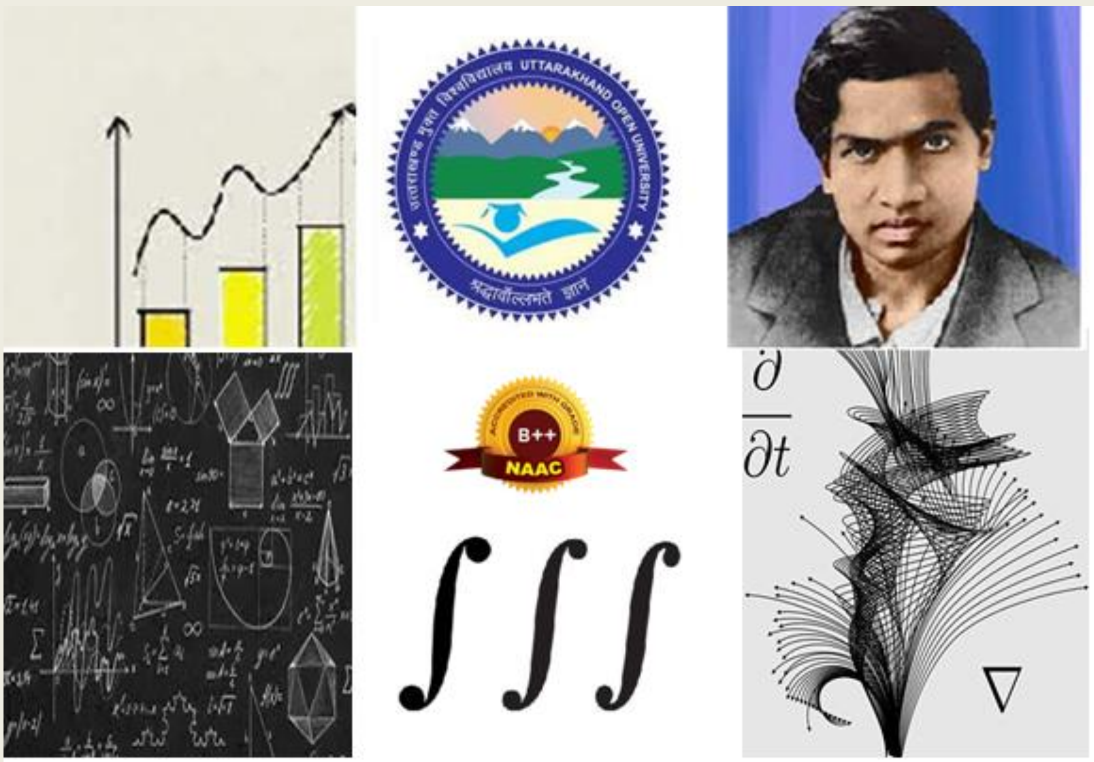
MASTER OF SCIENCE
MATHEMATICS
THIRD SEMESTER

MAT 602
FUNCTIONAL ANALYSIS



DEPARTMENT OF MATHEMATICS
SCHOOL OF SCIENCES
UTTARAKHAND OPEN UNIVERSITY
HALDWANI, UTTARAKHAND
263139

COURSE NAME: FUNCTIONAL ANALYSIS

COURSE CODE: MAT-602

Department of Mathematics
School of Science
Uttarakhand Open University
Haldwani, Uttarakhand, India,
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BOARD OF STUDIES - 2023

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COURSE CONTENTS

COURSE INFORMATION

The present self learning material **"Functional Analysis"** has been designed for M.Sc. (Third Semester) learners of Uttarkhand Open University, Haldwani. This course is divided into 14 units of study. This Self Learning Material is a mixture of Four Block.

The main objective of this course is to introduce the concepts of Functional Analysis simultaneously this course will provide the learners an opportunity to learn the basic Concepts and advanced concepts of Functional Analysis. The first block is Normed and Banach spaces it contains Basics: Basic definition and result of Metric Space, Basic definition and result of Vector Space. Normed linear space, Further properties of Normed space, Extended Real Number System, Holder's Inequality for finite sequence, Minkowski's Inequality for finite sequences, Holder's Inequality for infinite sequence, Minkowski's Inequality for infinite sequences, Continuous at a point, Cauchy Sequence, Completeness, Banach Space Finite dimensional Normed Spaces, Equivalent norms, Compactness, F. Riesz's Lemma. The second block is Linear Functional and Linear operator contains Linear operator, bounded and continuous linear operator, linear functional, linear functional of finite dimensional spaces, Normed space of operators and dual space. Third block is Inner product space and Hilbert space which is a mixture of Inner product spaces, Hilbert spaces and its example, Orthogonality, Orthonormal sets, Reisz Representation theorem, Legendre and Leguerre polynomial, Parsevals's theorem, the conjugate space of Hilbert space. Hilbert-Adjoint Operator, Self Adjoint, normal and unitary Operator, projection Operator and the last block is Fundamental Theorems for Normed and Banach Spaces which present the Zorn's lemma, Hahn-Banach theorem and its applications, Adjoint operator, Reflexive spaces, Category Theorem: Uniform Boundedness Theorem, Strong and Weak Convergence, Convergence of sequence operators and functional, Open Mapping Theorem, Closed Linear Operator. Closed Graph Theorem, Banach Fixed Point Theorem. On successful completion of this course, learners will be able to Appreciate how functional analysis uses and unifies ideas from different and diverse area of mathematics, Describe and apply fundamental theorems from the theory of normed and Banach spaces, including the Hahn-Banach theorem, parallelogram identity and Polarization identity and Recognize the role of Zorn's lemma.

Course Name: Functional Analysis Course Code MAT602

Credit: 4

Normed and Banach spaces

Basics: Basic definition and result of Metric Space, Basic definition and result of Vector Space. Normed linear space, Further properties of Normed space, Extended Real Number System, Holder's Inequality for finite sequence, Minkowski's Inequality for finite sequences, Holder's Inequality for infinite sequence, Minkowski's Inequality for infinite sequences, Continuous at a point, Cauchy Sequence, Completeness, Banach Space Finite dimensional Normed Spaces, Equivalent norms, Compactness, F. Riesz's Lemma.

Linear Functional and Linear operator

Linear operator, bounded and continuous linear operator, linear functional, linear functional of finite dimensional spaces, Normed space of operators and dual space.

Inner product space and Hilbert space

Inner product spaces, Hilbert spaces and its example, Orthogonality, Orthonormal sets, Reisz Representation theorem, Legendre and Leguerre polynomial, Parsevals's theorem, the conjugate space of Hilbert space. Hilbert-Adjoint Operator, Self Adjoint, normal and unitary Operator, projection Operator.

Fundamental Theorems for Normed and Banach Spaces

Zorn's lemma, Hahn-Banach theorem and its applications, Adjoint operator, Reflexive spaces, Category Theorem: Uniform Boundedness Theorem, Strong and Weak Convergence, Convergence of sequence operators and functional, Open Mapping Theorem, Closed Linear Operator. Closed Graph Theorem, Banach Fixed Point Theorem.

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SUGGESTED READINGS

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3. B. V. Limaye, (2014), *Functional Analysis*, New age International Private Limited.

BLOCK I: NORMED, BANACH SPACES

UNIT 1:

NORMED SPACE - I

CONTENTS:

1.1 INTRODUCTION

Before this unit we are assuming that learners are familiar with the basics of Real Analysis, Topology, Linear Algebra and Measure Theory. In functional analysis, a normed space is a vector space with a metric that allows the computation of vector length and distance between vectors. The present unit is devoted to the basic ideas of norm space.

Before this course we have studied about vector space and metric space. But there is no relation between the algebraic structure and the metric we cannot expect a useful and applicable theory that combines algebraic and metric concepts. To guarantee such a relation between "algebraic" and "geometric" properties of X we define on X a metric d in a special way as follows. We first introduce an auxiliary concept, the norm (definition below), which uses the algebraic operations of vector space. A large number of metric spaces in analysis can be regarded as normed spaces, so that a normed space is probably the most important kind of space in functional analysis, at least from the viewpoint of present-day applications.

1.2 OBJECTIVES

After studying this unit, learner will be able to

- **i.** Described the concept of *normed space.*
- **ii.** Evaluate the normed.
- **iii.** Problems and examples related to *normed space*.

1.3 BASICS

We first defined the basic definitions:

1.3.1 METRIC SPACE

Let $X \neq \emptyset$ be a set. A metric on the set X is essentially just a rule for calculating the distance between any two elements of X.

Metric space:

Let $X \neq \emptyset$ be a set then the metric on the set X is defined as a function $d: X \times X \rightarrow [0, \infty)$ such that the following conditions are satisfied

i. $d(x, y) \geq 0 \ \forall x, y \in X$ (self distance)

ii. $d(x, y) = 0$if and only if $x = y \forall x, y \in X$ (Positivity)

iii. $d(x, y) = d(y, x); \forall x, y \in X$ (Symmetry property)

iv. $d(x, y) \leq d(x, z) + d(z, y); \forall x, y, z \in X$ (Triangle inequality)

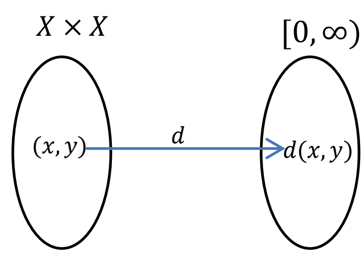


Fig.1.3.1. Metric Space

A metric space is an ordered pair (X, d) where X is a nonempty set and d is a metric on X.

Pseudo-metric:

Let $X \neq \emptyset$ be a set then thepseudo-metric on the set X is defined as a function $d: X \times X \rightarrow [0, \infty)$ such that it satisfies axioms $(M1), (M3)$and $(M4)$ of metric space and the axiom

$(M^{*}2) d(x, x) = 0 \ for \ all \ x.$

Every Metric is pseudo-metric but pseudo-metric need not to be metric.

NOTE:

Metric d is also known as distance function.

For a Pseudo-metric $x = y \Rightarrow d(x, y) = 0$ but converse may not be true.

Examples:

- Let X be any set and define the function $d : X \times X \rightarrow \mathbb{R}$ by

$$d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$$

Then d is a metric on X and called the discrete metric.

- The set $C[0,1]$ consisting of all real valued continuous functions defined on [0,1] with function d defined by $d(f,g) = \int_0^1 |f(x) - g(x)|\, dx \;\; for\ all\ f, g \in C[0,1]$. I a metric space.

Diameter:

Let (X,d) be a metric space and let Y be a non empty subset of X. Then the diameter of Y, denoted by $\delta(Y)$ be defined as

$$\delta(Y) = \sup\{d(x,y): x, y \in Y\}$$

i.e. diameter is the supremum of the set of all distance between point of Y.

Distance between point and set:

Let Y be a non empty subset of X and $p \in X$ then distance between point$p$ and Y is defined as

$$d(p,Y) = inf\ \{d(p,x): x\ in\ Y\}\ .$$

If $p \in Y$ then $d(p,Y) = 0$

Distance between two set:

Let Y_1andY_2 be a non empty subset of X then distance between Y_1andY_2 is defined as

$$d(Y_1, Y_2) = inf\ \{d(x,y): x\ in\ Y_1\ \ and\ y\ in\ Y_1\}$$

NOTE:

$d(Y_1, Y_2) \geq 0$and$d(Y_1, Y_2) \geq 0\ if\ and\ only\ if\ Y_1 \cap Y_2 \neq \emptyset$

$d(Y, \emptyset) = \infty$where$\emptyset$ is an empty set.

Bounded Metric spaces:

Let (X, d) be a metric space. Then X is said to be bounded if there exists $K \in \mathbb{R}^+$ such that $d(x, y) \leq K \; for \; all \; x, y \in X$.

Unbounded Metric spaces:

Let (X, d) be a metric space. Then X is said to be unbounded if it is not bounded.

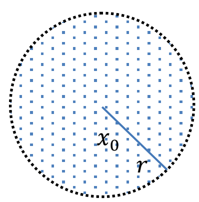
Open Sphere:

Let (X, d) be a metric space and let $x_0 \in X$. If r be any real number then the set $x \in X: d(x, x_0) < r$ is said to be open sphere or open ball.

Here x_0 is said to be centre of the open sphere and r is called the radius of the open sphere.

Open sphere of centrex_0 and radius r is denoted by $S(x_0, r)$.

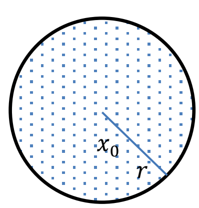
Therefore mathematically $S(x_0, r) = \{x \in X: d(x, x_0) < r\}$



***Open** sphere*

Closed Sphere:

Let (X, d) be a metric space and let $x_0 \in X$.Ifr be any real number then the set $S[x_0, r) = \{x \in X: d(x, x_0) \leq r\}$ is said to be closed sphere or closed ball.



Closed Sphere

NOTE:

- Sphere or open sphere or open ball or open cell or open disc are same.
- In the usual metric space $\mathbb{R}^n$, the open sphere $S(r, x_0)$ is circular disc $|x - x_0| < r$ and $x_0 \in \mathbb{R}^n$ and $r > 0$

Neighbourhood of a point in metric space:

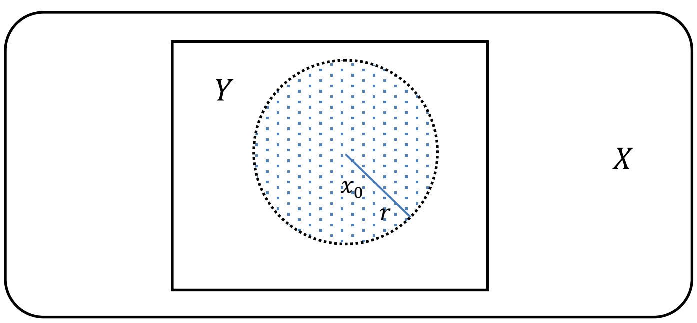
Let (X, d) be a metric space and $x_0 \in X$. A subset Y of X is said to be neighbourhood of a point $x_0$there exists $r > 0$ such that $S(x_0, r) \subseteq Y$.

Open sets in metric space:

Let (X, d) be a metric space. A subset Y of X is said to be open or d −open in Xif Y is neighbourhood of each of it points.

OR

Let (X, d) be a metric space. A subset Y of X is said to be open or d −open in Xiff for each $x \in Y$, there exists $r > 0$ such that $S(x, r) \subseteq Y$.



Open Set

Equivalent Metrics:

Let d and d' are two metrics on the same set X. Then d and d' are equivalent iff every d −open set is d' −open and every d' −open is d −open set .

Interior point:

Let (X, d) be a metric space and let Y be a subset of X. A point $x \in X$ is called an interior point of Y if there exists an open ball with centre x contained in Y, i.e.,

$$x \in S(x, r) \subseteq Y \; for \; some \; r > 0$$

Interior of Set:

The set of all interior points of Y is called the interior of Y and is denoted by $Int\ (Y)$or ° .

$$Int\ (Y) = \{x \in Y \; such \; that \; \in S(x, r) \subseteq Y \; for \; some \; r > 0\}$$

Exterior points:

Let (X, d) be a metric space and let Y be a subset of X. A point $x \in X$ is called a exterior point of Y if it is an interior point of the complement of Y i.e. Y^c.

Exterior of Set:

The set of all exterior points of Y is called the exterior of Y and is denoted by $ext\ (Y)$ or Y^e. i.e. $ext(A) = int(A^c)$

Frontier points:

Let (X, d) be a metric space and let Y be a subset of X. A point $x \in X$ is called a frontier point of Y if it is neither interior or nor exterior point of Y.

Frontier of Set:

The set of all frontier points of Y is called the frontier of Y and is denoted by $Fr\ (Y)$.

Boundary point:

Let (X, d) be a metric space and let Y be a subset of X. A point $x \in X$ is called a boundary point of Y if it is frontier point of Y and belong to Y.

Boundary of Set:

The set of all boundary points of Y is called the boundary of Y and is denoted by $b\ (Y)$.

Dense set:

Let (X, d) be a metric space and let Y_1 and Y_2 be subsets of X. Then Y_1 is said to be dense in Y_1 if $Y_2 \subseteq \overline{Y_1}$.

Everywhere Dense:

Let (X, d) be a metric space and let Y_1 be a subset of X. ThenY_1 is said to be dense in Xor everywhere dense if $\overline{Y_1} = X$.

Separable:

Let (X, d) be a metric space. X is said to be separable if it has a countable subset which is dense in X.

Nowhere Dense:

Let (X, d) be a metric space and let Y_1 be a subset of X. ThenY_1 is said to be nowhere dense in Xif interior of the closure of Y is empty.

Limit Point:

Let (X, d) be a metric space and let Y be a subset of X. A point $x \in X$ is called a limit point (an accumulation point) if every neighbourhood of x contains a point of Y distinct from x.

Derived Set:

The set of all limit points of Y is called the derived set of Y and denoted by $D(Y)$.

Adherent Point:

Let (X, d) be a metric space and let Y be a subset of X. A point $x \in X$ is called an adherent point of Y if every neighbourhood of x contains a point of Y(not necessarily distinct from x).

Adherence of Set:

The set of all adherent points of Y is called the adherence of Y. It is denoted by $Adh(Y)$.

Isolated points:

Let (X, d) be a metric space and let Y be a subset of X.A point $x \in X$ is called a islolated point of Y if $x \in X$ but not limit point of Y.

Closed Sets:

Let (X, d) be a metric space. A subset Y of X is said to be closed or d-closed if the compliment of Y is open.

OR

A subset Y of the metric space (X, d) is said to be closed if it contains each of its limit points, i.e.,$D(Y) \subseteq Y$.

Isometric mapping, isometric spaces:

Let $X = (X, d)$ and $\hat{X} = (\hat{X}, d)$ be metric spaces. Then:

i. A mapping T of X into $\hat{X}$ is said to be isometric or an isometry if T preserves distances, that is, if for all $x, y \in X$,

$$\hat{d}(Tx, Ty) = d(x, y),$$

where Tx and Ty are images of x and y, respectively.

ii. The space X is said to be isometric with the space $\hat{X}$ if there exists a bijective isometry of X onto $\hat{X}$.The spaces X and $\hat{X}$ are then called isometric spaces.

1.3.2 VECTOR SPACE

Definition- Let V be a nonempty set with two operations

(i) Vector addition: If any $u, v \in V$ then $u + v \in V$

(ii) Scalar Multiplication: If any $u \in V$ and $k \in F$ then $ku \in V$

Then V is called a vector space (over the field F) if the following axioms hold for any vectors if the following conditions hold

[S1] $(u + v) + w = u + (v + w)$ for any vectors $u, v, w \in V$

[S2] there exists a vector denoted by '0' in V, such that, for any $u \in V$,

$$u + 0 = 0 + u = u$$

Here '0' is called zero vector

[S3] for each $u \in V$ there exists a vector denoted by '$-u$' in V such that

$$u + (-u) = 0 = (-u) + u$$

Here '$-u$' is called additive inverse of vector 'u'

[S4] $u + v = v + u$ for any vectors $u, v \in V$

[P1] $k(u + v) = ku + kv$, for any $u \in V$ and for any scalar $k \in F$

[P2] $(k_1 + k_2)u = k_1 u + k_2 u$, for any $u \in V$ and for any scalar $k_1, k_2 \in F$

[P3] $(k_1 k_2)u = k_1(k_2 u)$, for any $u \in V$ and for any scalar $k_1, k_2 \in F$

[P4] $1.u = u$, for any $u \in V$ and for unit scalar $1 \in F$

The elements of the fiels F are called scalars and the elements of the vector space V are called vectors.

NOTE:

(i) The conditions $[S_1]-[S_4]$ concerned with additive structure of V and can be summarized by saying that V is a commutative group under addition.

(ii) The vector space V over the field F is denoted by $V(F)$.

1.4 NORMED SPACE

In this section we are defining definition of *normed space*. We first introduce subsidiary concept, the norm, which uses the algebraic operations of vector space. Then we use the norm to obtain a metric d that is of the desired kind. This idea gives to the concept of a normed space.

Let X be a vector space over scalar field K. A *norm* on a (real or complex) vector space X is a real-valued function on X ($\|x\|: X \to K$) whose value at an $x \in X$ is denoted by

$\|x\|$ (read "*norm* of x"),

and which has the properties:

($\boldsymbol{N1}$) $\quad \|x\| \geq 0 \forall\, x \in X$

($\boldsymbol{N2}$) $\quad \|x\| = 0 \Leftrightarrow x = 0, \forall\, x \in X$

($\boldsymbol{N3}$) $\quad \|\alpha x\| = |\alpha| \|x\| \forall\, \alpha \in K, \forall x \epsilon X$

($\boldsymbol{N4}$) $\quad \|x + y\| \leq \|x\| + \|y\| \;\forall x, y \epsilon X$

(Triangle inequality);

here x and y are arbitrary vectors in X and α is any scalar.

Whenever we are confronted with the problem of verifying whether given function defines a norm or not, the first three properties will be more or less obvious, and most of the effort, if any, would go in verifying this last statement, namely the triangle inequality. So, once a vector space with a norm would be called a normed linear space. A norm on X defines a metric d on X which is given by

$$d(x, y) = \|x - y\|, (x, y \in X) \dots\dots\dots\dots (1)$$

and is called the metric induced by the norm.

The normed space just defined is denoted by $(X, \|.\|)$ or simply by X.

- The norm is continuous, that is, $x \longrightarrow \|x\|$ is a continuous mapping of $(X, \|.\|)$ into $\mathbb{R}$.

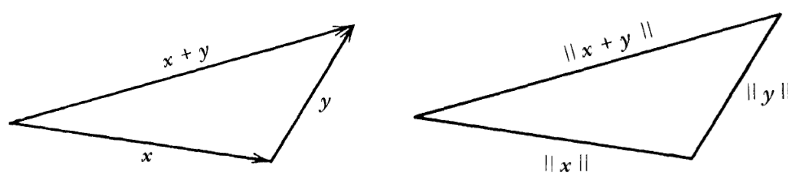


Fig 1.4.1

- The defining properties $\boldsymbol{N1}$ to $\boldsymbol{N4}$ of a norm are suggested and motivated by the length $|x|$, of a vector x in elementary vector algebra, so that in this case we can write $\|x\| = |x|$.
- The, $\boldsymbol{N1}$ and $\boldsymbol{N2}$ state that all vectors have positive lengths except the zero vector which has length zero.
- $\boldsymbol{N3}$ means that when a vector is multiplied by a scalar, its length is multiplied by the absolute value of the scalar.
- $\boldsymbol{N4}$ is explained in above figure. It means that the length of one side of a triangle cannot exceed the sum of the lengths of the two other sides.

- It is not difficult to conclude from $\boldsymbol{N1}$ to $\boldsymbol{N4}$ that (1) does define a metric.
- Hence normed spaces and Banach spaces are metric spaces.

1.4.1 EXAMPLES AND RESULTS

Example 1: $X = \mathbb{R}$ (set of reals). Define $\|x\| = |x|, \forall x \epsilon \mathbb{R}$.

Here $\| \quad \|: \mathbb{R} \to \mathbb{R}$,

$\forall\, x \in X$,

$|x| \geq 0$ … … ..(i)

$\forall\, x \in X, \forall\, \alpha \in \mathbb{R}$;

$\|\alpha x\| = |\alpha x| = |\alpha|\|x\| = |\alpha||x|$ … … … … ….(ii)

$\forall\, x, y \in X,$

$\|x + y\| = |x| + |y| = \|x\| + \|y\|$………………….(iii)

Again,

$\|x\| = 0 \Leftrightarrow |x| = 0$

$\Leftrightarrow x = 0.$

Thus $(\mathbb{R}, \|x\|)$ is a normed linear space.

Example 2: $X = \mathbb{C}$ (set of complex numbers).. $x = a + ib \,\epsilon\, \mathbb{C}$,

Define $\|x\| = \sqrt{a^2 + b^2} = |a + ib| = |x|$.

i. $\|x\| = \|(x_1 + ix_2)\| = \sqrt{{x_1}^2 + {x_2}^2} \geq 0.$

Therefore, $\forall\, x \in \mathbb{C}, \|x\| \geq 0$.

ii. $\forall\, x \in \mathbb{C}$ and $\forall\, \alpha \in \mathbb{C}$,

$$\|\alpha x\| = \|\alpha(x_1 + ix_2)\| = \|(\alpha x_1 + i\alpha x_2)\|$$

$$= \sqrt{(\alpha x_1)^2 + (\alpha x_2)^2}$$

$$= |\alpha|\sqrt{{x_1}^2 + {x_2}^2} = |\alpha|\|x\|.$$

iii. Let $z_1, z_2 \in \mathbb{C}$.

$\|z_1 + z_2\|^2 = |z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)}$

(Since $\|z\|^2 = z.\bar{z}$)

$\|z_1 + z_2\|^2 = |z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1} + \overline{z_2})$

[as $\overline{(z_1 + z_2)} = \overline{z_1} + \overline{z_2}$]

$\|z_1 + z_2\|^2 = (z_1 + z_2)(\overline{z_1} + \overline{z_2})$

$= z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2}$

$\|z_1 + z_2\|^2 = |z_1|^2 + |z_2|^2 + (z_1\overline{z_2} + z_2\overline{z_1})$

$= |z_1|^2 + |z_2|^2 + 2Re(z_1\overline{z_2})$ (as $\overline{z_1} + \overline{z_2} = 2Rez$)

$= |z_1|^2 + |z_2|^2 + 2|z_1||\overline{z_2}|$

$= |z_1|^2 + 2|z_1||\overline{z_2}| + |z_2|^2$ as $|z| = |\bar{z}|$

$(|z_1| + |z_2|)^2$

or, $\|z_1 + z_2\| \leq |z_1| + |z_2|$,

$= \|z_1\| + \|z_2\|$

$\|z_1 + z_2\| \leq \|z_1\| + \|z_2\|$.

iv. For all $z \in \mathbb{C}$ $\|z\| = \|x + iy\| = \sqrt{x^2 + y^2} = 0 \Leftrightarrow x = 0, y = 0 \Leftrightarrow z = 0 + i0 = 0$ (Triangle inequality holds).

Thus $(\mathbb{C}, \|x\|)$ is a normed linear space.

Example 3: Let $\mathbb{C}^n$ be the set of all $n -$tuples of complex numbers. For $x = (x_1, x_2, \dots \dots . x_n) \epsilon \mathbb{C}^n$; define $\|x\| = \sqrt{\sum_{i=1}^{n} |x_i|^2}$, then, $(\mathbb{C}^n, \|\quad\|)$ is a normed linear space.

Solution:

i. For all $x = (x_1, x_2, \dots \dots . x_n) \epsilon \mathbb{C}^n$,

$|x_i| \geq 0$ for all $1 \leq i \leq n$,

or, $|x_i|^2 \geq 0$, or, $\|x\| = \sqrt{\sum_{i=1}^{n} |x_i|^2} \geq 0$ or, $\|x\| \geq 0$.

ii. For all $x\epsilon\mathbb{C}^n$, for all $\alpha\epsilon\mathbb{C}$

$$\|\alpha x\| = \sqrt{\sum_{i=1}^{n}(\alpha x_i)^2} = |\alpha|\|x\|.$$

Therefore, $\|\alpha x\| = |\alpha|\|x\|$for all $x\epsilon\mathbb{C}^n, \forall\alpha\ \epsilon\ \mathbb{C}$.

iii. For all $x = (x_1, x_2, \dots \dots . x_n), y = (y_1, y_2, \dots \dots . y_n)\ \epsilon\mathbb{C}^n$;

$x + y = (x_1 + y_1, x_2 + y_2, \dots \dots . x_n + y_n), \epsilon\mathbb{C}^n$;

Then,

$$\|x + y\|^2 = \sum_{i=1}^{n}|x_i + y_i|^2 = \sum_{i=1}^{n}|x_i + y_i|\,|x_i + y_i|$$

$$\leq \sum_{i=1}^{n}|x_i + y_i|\,(|x_i| + |y_i|)[\text{Since, } |x_i + y_i| \leq |x_i| + |y_i|]$$

$$= \sum_{i=1}^{n}|x_i + y_i|\,|x_i| + \sum_{i=1}^{n}|x_i + y_i|\,|y_i|$$

$$\leq \|x + y\|\|x\| + \|x + y\|\|y\|[\sum_{i=1}^{n}|x_i + y_i| \leq \|x\| + \|y\|]$$

$$= \|x + y\|(\|x\| + \|y\|)$$

or, $\|x + y\|^2 \leq \|x + y\|(\|x\| + \|y\|)$

or, $\|x + y\| \leq \|x\| + \|y\|$. For $\|x + y\| \neq 0$.

iv. $\|x\| = \sum_{i=1}^{n}|x_i|^2 = 0 \Leftrightarrow x_i = 0,$ for all $1 \leq i \leq n$

$$\Leftrightarrow x = (x_1, x_2, \dots \dots . x_n) = (0,0,0 \dots \dots . .0, \dots \dots 0) = 0$$

$$\Leftrightarrow x = 0$$

1.4.2 PSEUDO NORM

Let X be a vector space over scalar field K ($\mathbb{R}$ or $\mathbb{C}$).

A function $\|\ \ \|$ on X into $\mathbb{R}$ is said to be a semi – *norm* or *Pseudo – norm* if

($\boldsymbol{N1}$) $\|x\| \geq 0, \forall\ x \in X$

($\boldsymbol{N3}$) $\|\alpha x\| = |\alpha|\|x\|, \forall\ \alpha \in K, \forall x\epsilon X$

($\boldsymbol{N4}$) $\|x + y\| \leq \|x\| + \|y\|, \forall x, y\epsilon X$

(Triangle inequality);

**It means that the in a *semi – norm* or *Pseudo – norm* by second property of norm may fail it implies norm may be zero for vectors other than the origin. C

- $\|x\| = |x_1|$ on $\mathbb{R}^2$. $\forall x = (x_1, x_2,) \in \mathbb{R}^2$
- If e_1, e_2 is a standard basis on $\mathbb{R}^2$, then define $\|x\| = |c_1 + c_2|$ where $x \in \mathbb{R}^2$ has the unique linear combination representation $x = c_1e_1 + c_2e_2$ where c_1 and c_2 are constant.

The examples are *semi - norm* or *Pseudo – norm*. We can show easily.

1.4.3 MAIN RESULTS

1. Proof that every *norm* is a *semi-norm* but converse is not necessarily.

Solution: Let $(X, \| \quad \|)$ be a normed linear space.

Then,

$(\boldsymbol{N1})$ $\quad \|x\| \geq 0 \forall\, x \in X$

$(\boldsymbol{N2})$ $\quad \|x\| = 0 \Leftrightarrow x = 0, \forall\, x \in X$

$(\boldsymbol{N3})$ $\quad \|\alpha x\| = |\alpha|\|x\| \forall\, \alpha \in K, \forall x \epsilon X$

$(\boldsymbol{N4})$ $\quad \|x + y\| \leq \|x\| + \|y\|\ \forall x, y \epsilon X$

(Triangle inequality);

As we know that by condition $(N1), (N3)$ and $(N4)$ the function $\|x\|: X \to \mathbb{R}$ is a *semi – norm*.

Consider$(\mathbb{R}^3, \| \quad \|)$ $\forall x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\|\boldsymbol{x}\| = |\boldsymbol{x_1}| + |\boldsymbol{x_2}|$.

$(\boldsymbol{N1})$. $\quad \forall x \in \mathbb{R}^3, |x_1| \geq 0, |x_2| \geq 0,\ |x_1| + |x_2| \geq 0$, so $\|x\| \geq 0$.

$(\boldsymbol{N3})$ $\quad \forall\, \alpha \in \mathbb{R}$,

$$\|\alpha x\| = \|\alpha(x_1, x_2, x_3)\| = \|\alpha x_1, \alpha x_2, \alpha x_3\|$$
$$= |\alpha x_1| + |\alpha x_2|$$
$$= |\alpha|(|x_1| + |x_2|) = |\alpha|(|x_1| + |x_2|)$$
$$= |\alpha| \, \|x\| \forall \alpha \in \mathbb{R}, \forall x \in \mathbb{R}^3.$$

$(\boldsymbol{N4})$ Let $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in \mathbb{R}^3$

$$x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$$
$$\|x + y\| = |x_1 + y_1| + |x_2 + y_2| \leq |x_1| + |y_1| + |x_2| + |y_2|$$
$$= (|x_1| + |x_2|) + (|y_1| + |y_2|) = \|x\| + \|y\|$$

or,

$$\|x + y\| \leq \|x\| + \|y\| \; \forall x, y\epsilon \in \mathbb{R}^3.$$

By, $(\boldsymbol{N1}), (\boldsymbol{N3})$ and $(\boldsymbol{N4})$ the function $\| \quad \|$ on $\mathbb{R}^3$ into $\mathbb{R}$ is semi-norm.

Now take $z = (0,0,1) \neq 0, z \in \mathbb{R}^3$.

But $\|z\| = |0| + |0| = 0$.

Therefore, $z \neq 0, \|z\| = 0$.

Thus $\| \quad \|: \mathbb{R}^3 \to \mathbb{R}$ is a not a norm. So, every semi-norm is not a norm.

2. Every normed linear space is a metric space. Converse is not necessary true.

Solution: Let $(X, \| \quad \|)$ be a normed linear space. Let $d: X \times X \longrightarrow \mathbb{R}$ be a function defined by $d(x, y) = \|x - y\| \forall x, y \in X$.

- **i.** $\forall x, y \in X, x - y \in X$. As $\|x - y\| \geq 0$,
  so, $d(x, y) = \|x - y\| \geq 0, \forall x, y \in X$
- **ii.** $d(x, y) = 0 \Leftrightarrow \|x - y\| = 0 \Leftrightarrow x - y = 0 \Leftrightarrow x = y$.
- **iii.** $d(x, y) = \|x - y\| = \|(-1)(y - x)\|$
  $= |-1|\|y - x\| = 1. \|y - x\| = d(y, x)$.
- **iv.** $d(x, y) = \|x - y\| = \|(x - z) + (z - y)\|$
  $\leq \|x - z\| + \|z - y\|$

$$= d(x,z) + d(z,y), \forall x, y \in X.$$

Therefore, (X, d) is a metric space.

Conversely, let $x = (x_n)_{n=1}^{\infty}$, $y = (y_n)_{n=1}^{\infty} \in \mathbb{C}^N$, where $\mathbb{C}^N$ is the set of all sequence of complex numbers.

Define $d(x,y) = \sum_{i=1}^{n} \frac{1}{2^i} \left[\frac{|x_i - y_i|}{1+|x_i - y_i|}\right]$

i. $\forall\, x = (x_n)_{n=1}^{\infty}$, $y = (y_n)_{n=1}^{\infty}$,

$|x_i - y_i| \geq 0 \forall i$

or, $\frac{1}{2^i} \left[\frac{|x_i - y_i|}{1+|x_i - y_i|}\right] \geq 0.$

Therefore, $\sum_{i=1}^{n} \frac{1}{2^i} \left[\frac{|x_i - y_i|}{1+|x_i - y_i|}\right] \geq 0$. So, $d(x,y) \geq 0 \forall\, x, y \in \mathbb{C}^N$.

ii. $d(x,y) = 0$

$$\Leftrightarrow \sum_{i=1}^{n} \frac{1}{2^i} \left[\frac{|x_i - y_i|}{1+|x_i - y_i|}\right] = 0$$

$\Leftrightarrow \frac{1}{2^i} \left[\frac{|x_i - y_i|}{1+|x_i - y_i|}\right] = 0$

$\Leftrightarrow x_i - y_i = 0$

$\Leftrightarrow x_i = y_i \forall i$

Therefore, $x = y$

$d(x,y)$=0 $\Leftrightarrow$ $x = y$

iii. $d(x,y) = \sum_{i=1}^{n} \frac{1}{2^i} \left[\frac{|x_i - y_i|}{1+|x_i - y_i|}\right] = \sum_{i=1}^{n} \frac{1}{2^i} \left[\frac{|y_i - x_i|}{1+|x_i - y_i|}\right] =$

$d(y,x)\ \forall\, x, y \in \mathbb{C}^N$.

iv. Let $z = (z_n)_{n=1}^{\infty} \in \mathbb{C}^N$ then,

$$d(x,y) = \sum_{i=1}^{n} \frac{1}{2^i} \left[\frac{|(x_i - z_i) + (z_i - y_i)|}{1 + |x_i - y_i|}\right]$$

$$\leq \sum_{i=1}^{n} \frac{1}{2^i}\left[\frac{|x_i - z_i|}{1 + |x_i - y_i|}\right] + \sum_{i=1}^{n} \frac{1}{2^i}\left[\frac{|z_i - y_i|}{1 + |z_i - y_i|}\right]$$

Since $\frac{|a+b|}{1+|a+b|} \leq \frac{|a|}{1+|a+b|} + \frac{|b|}{1+|a+b|}$.

$d(x,y) \leq d(x,z) + d(z,y), \forall\ x, y, z \in \mathbb{C}^N$.

Therefore $(\mathbb{C}^N, d)$ is a metric space.

Now, $d(\alpha x, \alpha y) = \sum_{i=1}^{n} \frac{1}{2^i}\left[\frac{|\alpha x_i - \alpha y_i|}{1+|x_i - y_i|}\right] = \sum_{i=1}^{n} \frac{1}{2^i}\left[\frac{|\alpha||x_i - y_i|}{1+|x_i - y_i|}\right]$

$= |\alpha| \sum_{i=1}^{n} \frac{1}{2^i}\left[\frac{|x_i - y_i|}{1+|x_i - y_i|}\right] \neq |\alpha| d(x,y)$, where, $\alpha \neq \pm 1$.

Where $d(x,y) = |\alpha x - \alpha y| = |\alpha|\|x - y\|$.

Therefore, $(\mathbb{C}^N, d)$ is a metric space but not normed linear space.

So every metric space is not normed linear space.

Remark:

Whenever we are addressed with the problem of verifying whether given function defines a norm or not, the first three properties will be more or less obvious, and most of the effort, if any, would go in verifying the triangle inequality.

So, once a vector space with a norm would be called a normed linear space. So, given a normed linear space we can define a metric $d(x,y) = \|x - y\| \geq 0, \forall x, y \in X$ It is clear that $d(x,y)$ is non-negative and $d(x,y) = 0$ if and only if $x = y$. Now, by the triangle inequality, we get $d(x,y) \leq d(x,z) + d(z,y); \forall x, y, z \in X$ Therefore, the distance function d satisfies the usual triangle inequality for a metric; and that is why we have the same name for these two inequalities.

Therefore, automatically a normed linear space gets a topology defined by this norm which is a nice metric topology; and that is called the norm topology of this vector space.

SOME NORMS

S.No.	Space	Norm $\|\boldsymbol{x}\|$
1.	$\mathbb{R}^n$ and $\mathbb{C}^n$	$\left(\sum_{j=1}^{n}\lvert x_j\rvert^2\right)^{1/2}=\sqrt{\lvert x_1\rvert^2+\cdots\lvert x_n\rvert^2}$
2.	l^p	$\left(\sum_{j=1}^{\infty}\lvert x_j\rvert^p\right)^{1/p}$ where $1\le p<\infty$
3.	l^∞	$sup_j\lvert x_j\rvert$ if $p=\infty$
4.	$C[a,b]$	$max_{t\in j}\lvert x(t)\rvert$
5.	Set of all continuous real –valued functions on [0,1]	$\int_0^1 \lvert x(t)\rvert\, dt$

Note:

$l^p \subset l^{p'}$ if $l \le p \le p' \le \infty$.

Note:

$c = \{x \in l^\infty : (x(j))$ converges in $\mathbb{K}\}$.

$c_0 = \{x \in c : (x(j))$ converges to 0 in $\mathbb{K}\}$.

$c_{00} = \{x \in l^p\ all\ but\ finetely\ many x_j' sare\ 0\}, 1 \le p \le \infty$.

Note:

For $1 \le p < \infty$, by $L^p(E)$, we mean a collection of equivalence classes $[f]$ for which $|f|^p$ is integrable. Thus

$$f \in L^p(E) \Leftrightarrow \int_E |f|^p < \infty.$$

Sometimes we denote the collection of such functions by the symbol L^p.

Note:

A measurable function f on measurable set E is said to be an essentially bounded function if there exists $M_f > 0$ such that

$|f(x)| \leq M_f$ for all most all $x \in E$.

We define $L^\infty(E)$ to be the collection equivalence classes $[f]$ for which f is essentially bounded functions on E.

Therefore $f \in L^\infty(E) \Leftrightarrow$ there exists $M_f > 0$ such that $|f(x)| \leq M_f$ for almost all $x \in E$.

Note:

For E a measurable set, $1 \leq p < \infty$, and a function f in $L^p(E)$, we denote

$\|f\|_p := (\int_E |f|^p)^{1/p}$, and for $p = \infty, \|f\|_\infty = inf\ \{M_f > 0 :$

$|f(x)| \leq M_f$ for almost all $x \in E\}$.

Note:

For $1 \leq p \leq \infty$, $L^p(E)$ is a vector space over R.

1.4.4 IMPORTANT EXAMPLE

Problem1: The set $S(0;1) = \{x \in X: \left|\left|x\right|\right| = 1\}$ is known as unit sphere in norm linear space.

Show that in a vector space X, with different norms $S(0;1)$ can be different.

Solution: Consider $X = R^2$, with four different norms $: ||.||_1, ||.||_2, ||.||_4, ||.||_\infty$.

Under these four norms R^2, $S(0;1)$ are different. First consider $(R^2, ||.||_1)$.

Then for $(x,y) \in S(0;1) \subseteq (R^2, ||.||_1)$ gives

$$||(x,y)||_1 = 1 \Rightarrow |x| + |y| = 1 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

which represents four line segments simultaneously as follows:

(i) When $x > 0$ and $y > 0$ (i.e. in first quadrant of R^2 plane) equation (1) becomes

$$x + y = 1.$$

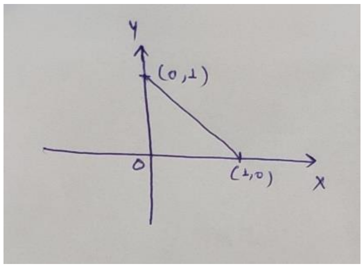


Fig: 1.4.4.1

(ii) When $x < 0$ and $y > 0$ (i.e. in second quadrant of R^2 plane) equation (1) becomes

$$-x + y = 1.$$

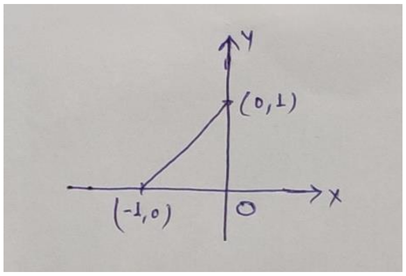


Fig: 1.4.4.2

(iii) When $x < 0$ and $y < 0$ (i.e. in third quadrant of R^2 plane) equation (1) becomes

$$-x - y = 1.$$

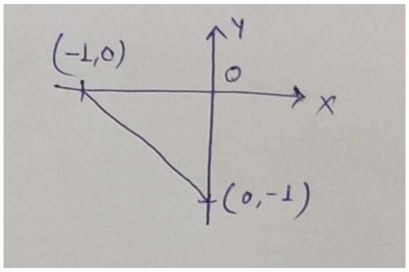


Fig: 1.4.4.3

(iv) When $x > 0$ and $y < 0$ (i.e. in first quadrant of R^2 plane) equation (1) becomes

$$x - y = 1.$$

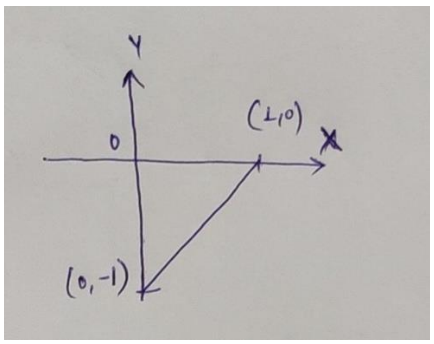


Fig: 1.4.4.4

Combining all of the above four cases the unit sphere $S(0;1)$ in $(R^2, ||.||_1)$ is represented in the following figure:

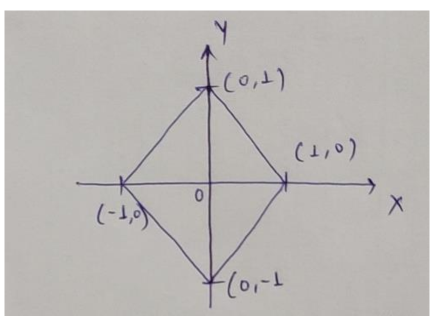


Fig: 1.4.4.5

Now take$(R^2, ||.||_2)$. Then for $(x, y) \in S(0; 1) \subseteq (R^2, ||.||_2)$ gives

$$\left||(x,y)|\right|_2 = 1 \Rightarrow \sqrt{x^2+y^2} = 1 \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

which is a equation of circle with center (0,0) and radius 1. Hence the unit sphere $S(0; 1)$ in $(R^2, ||.||_2)$ is represented in the following figure:

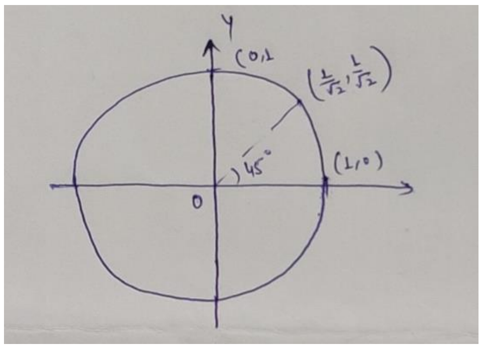


Fig: 1.4.4.6

Now consider$(R^2, ||.||_\infty)$. Then for $(x, y) \in S(0; 1) \subseteq (R^2, ||.||_\infty)$ gives

$$\left||(x,y)|\right|_\infty = 1 \Rightarrow \max\{|x|, |y|\} = 1 \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

which represents four line segments simultaneously. By the definition of maximum

$$\max\{|x|, |y|\} = \begin{cases} |x|, if\ |x| \geq |y| \\ |y|, |y| < |x|. \end{cases}$$

And the condition $|x| \geq |y|$ gives $\frac{|y|}{|x|} \leq 1$. This implies that $|tan\theta| \leq 1$, where θ is defined as in the following figure. Which further gives $\frac{-\pi}{4} \leq \theta \leq \frac{\pi}{4}$ and $\frac{\pi}{2} + \frac{\pi}{4} \leq \theta \leq \pi + \frac{\pi}{4}$.

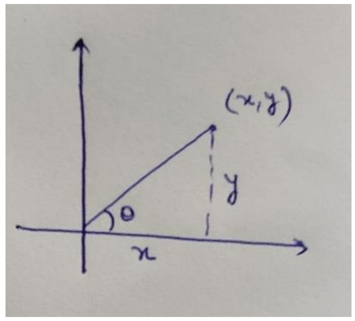


Fig: 1.4.4.7

Hence from equation (3) and by the definition of $\max\{|x|,|y|\}$ we get $\max\{|x|,|y|\} = |x| = 1$, when $\frac{-\pi}{4} \leq \theta \leq \frac{\pi}{4}$ and $\frac{\pi}{2} + \frac{\pi}{4} \leq \theta \leq \pi + \frac{\pi}{4}$. These conditions gives two line segments, which are represented in following figure:

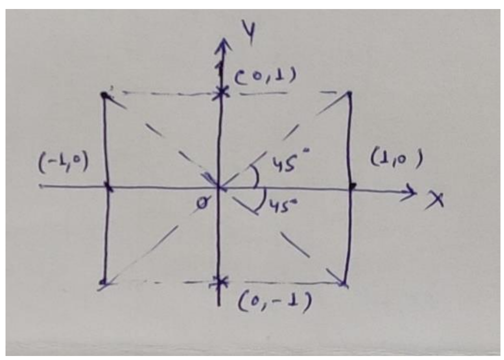


Fig: 1.4.4.8

Similarly $\max\{|x|,|y|\} = |y| = 1$, when $|y| < |x|$ is represented in following figure:

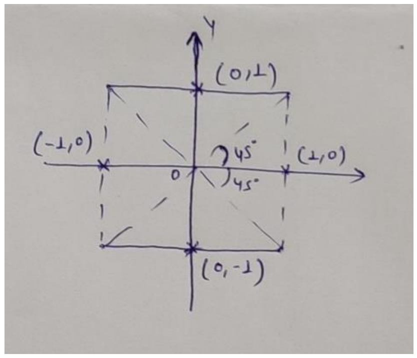


Fig: 1.4.4.9

Combining above two cases, $(x, y) \in R^2$ such that $\max\{|x|, |y|\} = 1$ is represented by following figure:

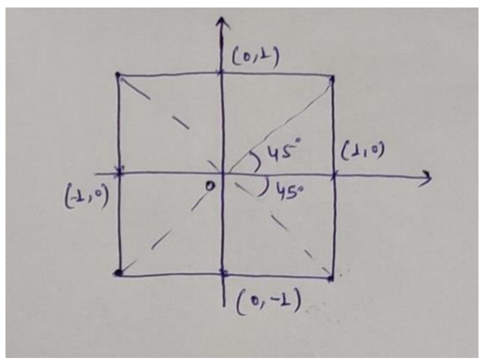


Fig: 1.4.4.10

Now consider R^2 with $||.||_4$Then for $(x, y) \in S(0; 1) \subseteq (R^2, ||.||_4)$ gives

$$\left||(x, y)|\right|_4 = 1$$

$$\Rightarrow (|x|^4 + |y|^4)^{\frac{1}{4}} = 1$$

$$\Rightarrow |x|^4 + |y|^4 = 1$$

$$\Rightarrow |y|^4 = 1 - |x|^4$$

$$\Rightarrow y = \pm\sqrt[4]{1 - |x|^4}.$$

The above equation is represented in R^2 as in the following figure:

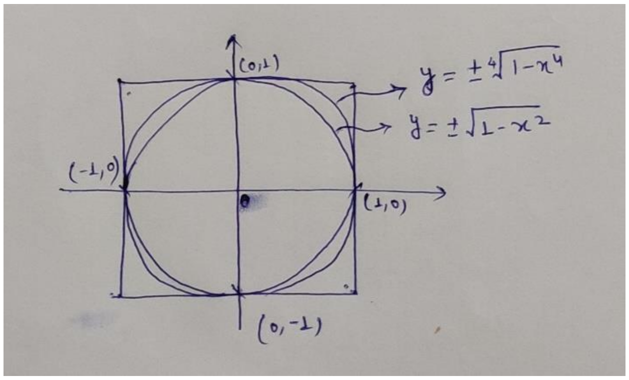


Fig: 1.4.4.11

Problem2: In a norm linear space $(X, ||.||)$, show that closed unit ball $B(0;1) = \{x \in X: ||x|| \leq 1\}$ is a convex set.

Solution: Let $x, y \in B(0;1)$ and $0 \leq \alpha \leq 1$. Then

$$||\alpha x + (1-\alpha)y|| \leq ||\alpha x|| + ||(1-\alpha)y||$$

$$= |\alpha|.||x|| + |(1-\alpha)|.||y|| \ldots\ldots\ldots\ldots\ldots.(4)$$

Since $x, y \in B(0;1)$and $0 \leq \alpha \leq 1$, therefore $||x|| = 1 = ||y||$ and $|\alpha| = \alpha$, $|1-\alpha| = (1-\alpha)$ Then inequality (4) becomes

$$||\alpha x + (1-\alpha)y|| \leq |\alpha|.1 + |(1-\alpha)|.1 = \alpha + 1 - \alpha = 1.$$

This implies that $\alpha x + (1-\alpha)y \in B(0;1)$. And hence $B(0;1)$is a convex set in $(X, ||.||)$.

Problem 3: Using the above problem, show that: in R^2, the mapping $\phi(x,y) = (\sqrt{x} + \sqrt{y})^2$, does not define a norm.

Solution: Assume that the mapping $\phi(x,y) = (\sqrt{x} + \sqrt{y})^2$ defines a norm on R^2. Then by above problem $B(0;1)$is a convex set. Consider $(1,0), (0,1) \in B(0;1)$ and $\alpha = 1/2$. Then

$$\alpha.(1,0)+(1-\alpha).(0,1)=\frac{1}{2}.(1,0)+\left(1-\frac{1}{2}\right).(0,1)=\left(\frac{1}{2},0\right)+\left(0,\frac{1}{2}\right)$$
$$=\left(\frac{1}{2},\frac{1}{2}\right).$$

And $\phi\left(\frac{1}{2},\frac{1}{2}\right)=\left(\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{2}}\right)^2=\left(2\times\sqrt{\frac{1}{2}}\right)^2=4\times\frac{1}{2}=2.$ Hence $\left(\frac{1}{2},\frac{1}{2}\right)=\frac{1}{2}.(1,0)+\left(1-\frac{1}{2}\right).(0,1)\notin B(0;1)$. Thus a convex combination of $(1,0)\&(0,1)$ does not belongs to $B(0;1)$. Which contradicts the fact that $B(0;1)$is a convex set. Hence our assumption is wrong, i.e. $\phi(x,y)=(\sqrt{x}+\sqrt{y})^2$, does not define a norm on R^2.

Bounded Set: A subset M in normed space X is bounded if and only if there is a positive number c such that $\|x\|\leq c$ for every $x\in M$.

1.5 SUMMARY

This unit we have start from some basic definitions (metric space, vector space). After that we have defined the normed space (Let X be a vector space over scalar field K. A *norm* on a (real or complex) vector space X is a real-valued function on X ($\|x\|: X\to K$) whose value at an $x\in X$ is denoted by$\|x\|$ (read “*norm* of x"),and which has the four properties) then Examples defined, after that Semi-Norm(Let X be a vector space over scalar field K ($\mathbb{R}$ or $\mathbb{C}$). A function $\|\ \ \|$ on X into $\mathbb{R}$ is said to be a semi – *norm* or *Pseudo* – *norm* if $\|x\|$ (read “*norm* of x"),and which has the three properties) and Main Results defined.

1.6 GLOSSARY

i. **Set:** Any well-defined collection of objects or numbers are referred to as a set.

ii. **Interval:** An open interval does not contain its endpoints, and is indicated with parentheses. $(a,b) =]a,b[= \{x \epsilon \mathbb{R}: a < x < b\}$. A closed interval is an interval which contain all its limit points, and is expressed with square brackets. $[a,b] = [a,b] = \{x \epsilon \mathbb{R}: a \leq x \leq b\}$. A half-open interval includes only one of its endpoints, and is expressed by mixing the notations for open and closed intervals. $(a,b] =]a,b] = \{x \epsilon \mathbb{R}: a < x \leq b\}$. $[a,b) = [a,b[= \{x \epsilon \mathbb{R}: a \leq x < b\}$.

iii. **Ordered Pairs:** An ordered pair (a, b) is a set of two elements for which the order of the elements is of significance. Thus $(a,b) \neq (b,a)$ unless $a = b$. In this respect (a, b) differs from the set $\{a, b\}$.Again $(a,b) = (c,d) \Leftrightarrow a = c$ and $b = d$ .If X and Y are two sets, then the set of all ordered pairs (x, y), such that $x \in X$ and $y \in Y$ is called Cartesian product of X and Y.

iv. **Relation:** A subset R of $X \times Y$ is called relation of X on Y. It gives a correspondence between the elements of X and Y. If (x, y) be an element of R, then y is called image of x. A relation in which each element of X has a single image is called a function.

v. **Function:** Let X and Y are two sets and suppose that to each element x of X corresponds, by some rule, a single element y of Y. Then the set of all ordered pairs (x, y) is called function.

vi. **Variable:** A symbol such as x or y, used to represent an arbitrary element of a set is called a variable.

vii. **Metric space:** Let $X \neq \emptyset$ be a set then the metric on the set X is defined as a function $d: X \times X \rightarrow [0, \infty)$ such that some conditions are satisfied.

viii. **Vector space: -** Let V be a nonempty set with two operations

(i) **Vector addition:** If any $u, v \in V$ then $u + v \in V$

(ii) **Scalar Multiplication:** If any $u \in V$ and $k \in F$ then $ku \in V$

Then V is called a vector space (over the field F) if the following axioms hold for any vectors if the some conditions hold.

CHECK YOUR PROGRESS

Fill in the Blanks:

1. norm on a (real or complex) vector space X is a........... on X.

2. The norm is...... that is, $x \longrightarrow \|x\|$ is a continuous mapping of $(X, \|.\|)$ into $\mathbb{R}$.

True/False

3. Proof that every *semi -norm* is a *norm.* True/False.

4. Every normed linear space is a metric space. Converse is not necessary true. True/False.

5. Which of the following statements are true?

i. $l_1 \subset l_2$

ii. $l_2 \subset c_0$

iii. $l_2 \subset l_1$

6. Which of the following is not a linear space over $\mathbb{C}$?

i. The set of all convergent sequences in $\mathbb{C}$.

ii. The set of all bounded sequences in $\mathbb{C}$.

iii. The set of all sequences in $\mathbb{C}$ that converges to 0.

iv. The set of all sequences in $\mathbb{C}$ that converges to a real number.

7.

Which of the following denotes the space of all bounded scalar sequences?

(a) c

(b) ℓ_∞

(c) ℓ_p

(d) s

8.

If $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on a linear space E, then $\|\cdot\|_1$ is stronger than $\|\cdot\|_2$ if and only if :

(a) $\exists\ C > 0$ such that $\|x\|_2 \leq C\|x\|_1$, for all $x \in E$.

(b) $\exists\ C > 0$ such that $\|x\|_1 \leq C\|x\|_2$, for all $x \in E$.

(c) $\exists\ 0 < C < 1$ such that $\|x\|_2 \leq C\|x\|_1$, for all $x \in E$.

(d) $\exists\ 0 < C < 1$ such that $\|x\|_1 \leq C\|x\|_2$, for all $x \in E$.

9.

Let $(E, \|\cdot\|)$ be a normed space and let d be the metric induced by the norm on E. If $x, y \in E$ and if $d(x, y) = r$, then which of the following is false?

(a) $d(x + z, y + z) = r$, for any $z \in E$.

(b) $d(rx, ry) = r^2$

(c) $d(ax, ay) = |a|r$, for any scalar a.

(d) $d(rx + y, ry + x) = (r - 1)r$.

10.

Which of the following linear space is infinite dimensional?

(a) $\mathbb{R}$ over $\mathbb{Q}$

(b) $\mathbb{Q}$ over $\mathbb{Q}$

(c) $\mathbb{C}$ over $\mathbb{C}$

(d) $\mathbb{C}$ over $\mathbb{R}$

1.7 REFERENCES

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1.8 SUGGESTED READINGS

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3. B. V. Limaye, (2014), *Functional Analysis*, New age International Private Limited.

1.9 TERMINAL QUESTIONS

1. Prove that with $d(x, y) = |x - y|$, the absolute value of the difference $x - y$, for each $x, y \in \mathbb{R}$, $(\mathbb{R}, d)$ is a metric space.

2. Let $V = \{(x_1, x_2) : x_1, x_2 \in \mathbb{R}\}$. For $(x_1, x_2), (y_1, y_2) \in S$and $c \in \mathbb{R}$, define $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 - y_2)$ and $c(x_1, x_2) = (cx_1, cx_2)$. Prove that S is not a vector space.

3. Show that $(\mathbb{R}^n, \| \quad \|)$ is a normed linear space.

4. Show that the set of all real numbers, with the usual addition and multiplication, constitutes a one-dimensional real vector space, and the set of all complex numbers constitutes a one-dimensional complex vector space.

5. Show that if d is a metric on a vector space $X \neq 0$ which is obtained from a norm, and $\hat{d}$ is defined by $d(x, x) = 0, \hat{d}(x, y) = d(x, y) + 1, x \neq y$ show that $\hat{d}$ cannot be obtained from a norm.

6. Show that the norm $\|x\|$ of x is the distance from x to 0.

1.10 ANSWERS

CHECK YOUR PROGRESS

1. Real-valued function.

2. Continuous.

3. False

4. True

5. i and ii

6. d

7. b

8. a

9. d

10. a

UNIT 2:

NORMED SPACE-II

CONTENTS:

2.1 INTRODUCTION

In previous unit we have de Described the concept of *normed space*, *e*valuate the normed and describe the problems and examples related to *normed space*. In present unit first we are defining extended real number system . In mathematics, the extended real number system is obtained from the real number system $\mathbb{R}$ by adding two infinity elements: $+\infty,$ and $-\infty$.

The extended real number system is denoted $\mathbb{R}^* = \mathbb{R} \cup \{+\infty, -\infty\}$. After this Hölder's inequality defined in a simple manner. In Mathematics, Hölder's inequality, named after Otto Hölder, is a fundamental inequality between integrals and an indispensable tool for the study of L^p spaces. A complete study of Hölder's inequality is explaining here. After this Minkowski inequality explained here Minkowski inequality establishes that the L^p spaces are normed vector spaces.

2.2 OBJECTIVES

After studying this unit, learner will be able to

- **i.** Analyze the concept of extended real number system
- **ii.** Describe the Holder's Inequality for finite sequence
- **iii.** Defined the concept of Minkowski's Inequality for finite sequences
- **iv.** Discuss the Holder's Inequality for infinite sequence
- **v.** Explained the detailed concept regarding Continuity in Normed Linear Space

2.3 EXTENDED REAL NUMBER SYSTEM

Let $\mathbb{R}$ = set of real numbers = {all rationals, all irrationals}

$$\mathbb{R}^* = \mathbb{R} \cup \{+\infty, -\infty\}$$

Then $\mathbb{R}^*$ is called extended real number system.

Definition:

Let p be an extended real number such that $p \geq 1$.

An extended real number q is called conjugate index of p if,

i. $\frac{1}{p} + \frac{1}{q} = 1$, when $1 < p < \infty$

ii. $q = \infty$ when $p = 1$

iii. $q = 1$ when $p = \infty$

Example:

i. Let $p = 4$, then $\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow \frac{1}{q} = 1 - \frac{1}{p} = 1 - \frac{1}{4} = \frac{3}{4}$.

$$q = \frac{4}{3}$$

ii. Let $p = 3$, then $\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow \frac{1}{q} = 1 - \frac{1}{p} = 1 - \frac{1}{3} = \frac{2}{3}$

$$q = \frac{3}{2}$$

iii. Let $p = 3$, then $\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow \frac{1}{q} = 1 - \frac{1}{p} = 1 - \frac{1}{2} = \frac{1}{2}$

$$q = \frac{2}{1}$$

Remark:

i. $1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1$, then $1 < q < \infty$.

ii. By symmetry of definition, if q is the conjugate index of p then p is also conjugate index of q. Thus, p and q are conjugated indices of each other.

Lemma1 [Young's inequality]:

Let p be a $+ve$ real number such that $1 < p < \infty$, and let q be the conjugate index of $p, (1 < q < \infty)$.

Let a and b be two positive real numbers, then,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

……………………..[1]

Proof: Define, $f(t) = t^\alpha - \alpha t + \alpha - 1, \forall t \geq 0.$

Therefore, $f'(t) = \alpha t^{\alpha-1} - \alpha = \alpha\left[\frac{1}{t^{1-\alpha}} - 1\right]$

(Since $1 - \alpha > 0 \forall t \geq 0$).

So, $f'(t) \geq 0 \forall t \epsilon [0,1]$

and $f'(t) \leq 0 \ \forall t \epsilon (1, \infty)$.

So by Lagrange Mean Value Theorem, $f(t)$ is monotonic increasing in $0 \leq t \leq 1$ and monotonic decreasing in $[1, \infty)$.

Therefore, $f(t) \leq f(1) \ \forall t \epsilon [0, \infty)$.

For all $t \epsilon [0, \infty), f(1) = 0$.

$$t^\alpha - \alpha t + \alpha - 1 \leq 0 \ldots\ldots\ldots (\boldsymbol{i})$$

Given result is trivially satisfied if $a = 0$, or $b = 0$.

Now, let $a \neq 0, b \neq 0$ and $\alpha = \frac{1}{p}, t = a/b$

By (i) $(a/b)^{\frac{1}{p}} - \frac{1}{p}(a/b) + \frac{1}{p} - 1 \leq 0$

or, $(a/b)^{\frac{1}{p}} - \frac{1}{p}(a/b) \leq 1 - \frac{1}{p}$

or, $(a/b)^{\frac{1}{p}} - \frac{1}{p}(a/b) \ \leq \frac{1}{q}$, as $\frac{1}{q} = 1 - \frac{1}{p}$

or, $(a)^{\frac{1}{p}}(b)^{1-\frac{1}{p}} - a.\frac{1}{p} \leq \ b.\frac{1}{q}$,[multiplying by b].

or, $(a)^{\frac{1}{p}}(b)^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}$.

Replacing a by a^p and b by b^q in the above inequality, we have,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

2.4 HOLDER'S INEQUALITY FOR FINITE SEQUENCES

Let $x = (x_i)_{i=1}^n, y = (y_i)_{i=1}^n \in \mathbb{C}^n$.

Define, $\|\boldsymbol{x}\|_p = \left[\sum_{i=1}^n |x_i|^p\right]^{1/p}$, for $p > 1$.

Then, $\sum_{i=1}^n |x_i y_i| \leq \left[\sum_{i=1}^n |x_i|^p\right]^{1/p} \left[\sum_{i=1}^n |y_i|^q\right]^{1/q}$.

……………………………………….. [2]

Let $p > 1$ and define q by $\frac{1}{p} + \frac{1}{q} = 1$ p and q are called conjugate exponents.

$\sum_{i=1}^n |x_i y_i| \leq \|x\|_p \|y\|_q$.

Proof: Let $x = (x_1, x_2, \dots x_n)$, $y = (y_1, y, \dots y_n)$.

Case I: If $x = 0$ or $y = 0$, then the inequality is trivially satisfied.

Case II: If $x \neq 0$ or $y \neq 0$.

After using equation (1), $a_i b_i \leq \frac{a_i{}^p}{p} + \frac{b_i{}^q}{q}, a_i, b_i > 0$.

We take $a_i = \frac{|x_i|}{\|x\|_p}, b_i = \frac{|y_i|}{\|y\|_q}, p > 1, q > 1$

[Since $\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow \frac{1}{q} = 1 - \frac{1}{p} \Rightarrow q = \frac{1}{1-\frac{1}{p}} > 1$].

Thus, the above inequality becomes,

$$\frac{|x_i|}{\|x\|_p}\frac{|y_i|}{\|y\|_q} \leq \frac{1}{p}\frac{|x_i|^p}{\|x\|_p^p} + \frac{1}{q}\frac{|y_i|^q}{\|y\|_q^q}$$

or, $\sum_{i=1}^{n} \frac{|x_i|}{\|x\|_p}\frac{|y_i|}{\|y\|_q} \leq \frac{1}{p}\frac{\sum_{i=1}^{n}|x_i|^p}{\|x\|_p^p} + \frac{1}{q}\frac{\sum_{i=1}^{n}|y_i|^q}{\|y\|_q^q}$

$$= \frac{1}{p}\frac{\|x\|_p^p}{\|x\|_p^p} + \frac{1}{q}\frac{\|y\|_q^q}{\|y\|_q^q}$$

$$= \frac{1}{p} + \frac{1}{q} = 1$$

$$\sum_{i=1}^{n} \frac{|x_i|}{\|x\|_p}\frac{|y_i|}{\|y\|_q} = 1.$$

Or, $\sum_{i=1}^{n}|x_i|\,|y_i| \leq \|x\|_p\|y\|_q.$

Or, $\sum_{i=1}^{n}|x_i y_i| \leq \|x\|_p\|y\|_q.$

2.5 MINKOWSKI'S INEQUALITY FOR FINITE SEQUENCES

Let $x = (x_i)_{i=1}^{n}, y = (y_i)_{i=1}^{n} \in l_p^n; p \geq 1,$

$l_p^n = \{(x_i)_{i=1}^{n} \in \mathbb{C}^n : \sum_{i=1}^{n}|x_i|^p < \infty\}.$

Then, $\|\boldsymbol{x} + \boldsymbol{y}\|_p \leq \|\boldsymbol{x}\|_p + \|\boldsymbol{y}\|_p$.

i.e. $[\sum_{i=1}^{n}|x_i + y_i|^p]^{1/p} \leq [\sum_{i=1}^{n}|x_i|^p]^{1/p} + [\sum_{i=1}^{n}|y_i|^p]^{1/p}$**.....[3]**

$$\|\boldsymbol{x} + \boldsymbol{y}\|_p \leq \|\boldsymbol{x}\|_p + \|\boldsymbol{y}\|_p$$

Proof. For $p = 1$, $\sum_{i=1}^{n}|x_i + y_i| \leq \sum_{i=1}^{n}(|x_i| + (|y_i|)$

Since $|x_i + y_i| \leq |x_i| + |y_i| \forall i$

$$= \sum_{i=1}^{n}|x_i| + \sum_{i=1}^{n}|y_i|$$

$$= \|x\|_1 + \|y\|_1$$

$\|x + y\|_1 \leq \|x\|_1 + \|y\|_1.$

Hence given result is true for $p = 1$.

For $p > 1$, $\|x + y\|_p^p = \sum_{i=1}^{n} |x_i + y_i|^p$

$$=\sum_{i=1}^{n} |x_i + y_i|\, |x_i + y_i|^{p-1}$$

$$\leq \sum_{i=1}^{n} (|x_i| + |y_i|)\, |x_i + y_i|^{p-1}$$

Since $|x_i + y_i| \leq |x_i| + |y_i| \forall i$

$$= \sum_{i=1}^{n} |x_i|\, |x_i + y_i|^{p-1} + \sum_{i=1}^{n} |y_i|\, |x_i + y_i|^{p-1}$$

………………………..[4]

At first we shall show that

$$(|x_i + y_i|^{p-1})_{i=1}^{n} \in L_n^q$$

$$\sum_{i=1}^{n} (|x_i + y_i|^{p-1})^q = \sum_{i=1}^{n} |x_i + y_i|^{pq-q}$$

[Since $\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow p + q = pq \Rightarrow p = pq - q$].

$$\sum_{i=1}^{n} (|x_i + y_i|^{p-1})^q = \sum_{i=1}^{n} |x_i + y_i|^p < \infty (\because x, y \in L_n^p \Rightarrow x + y \in L_n^p)$$

Applying Holder's inequality in equation [4], we have,

$$\|x + y\|_p^p \leq \left[\sum_{i=1}^{n} |x_i|^p\right]^{1/p} \left[\sum_{i=1}^{n} |x_i + y_i|^q\right]^{1/q}$$

$$+ \left[\sum_{i=1}^{n} |y_i|^p\right]^{1/p} \left[\sum_{i=1}^{n} |x_i + y_i|^q\right]^{1/q}$$

$$= \|\boldsymbol{x}\|_p \|x + y\|_p^{p/q} + \|\boldsymbol{y}\|_p \|x + y\|_p^{p/q}$$

Since,

$$\left[\sum_{i=1}^{n} (|x_i + y_i|)^{pq-q}\right]^{1/q} = \left[\sum_{i=1}^{n} (|x_i + y_i|)^{p}\right]^{1/q} = \left[\sum_{i=1}^{n} (|x_i + y_i|)\right]^{p/q}$$

Or, $\|x+y\|_p^p \le \|\boldsymbol{x}\|_p \|x+y\|_p^{p/q} + \|\boldsymbol{y}\|_p \|x+y\|_p^{p/q}$

$$= \left(\|\boldsymbol{x}\|_p + \|\boldsymbol{y}\|_p\right)\|x+y\|_p^{p/q}$$

$$\|x+y\|_p^{p-\frac{p}{q}} \le \|\boldsymbol{x}\|_p + \|\boldsymbol{y}\|_p$$

[Since $\frac{1}{p}+\frac{1}{q}=1 \Rightarrow p+q=pq \Rightarrow q=pq-p \;\Rightarrow 1=p-\frac{p}{q}$]

$$\|\boldsymbol{x}+\boldsymbol{y}\|_p \le \|\boldsymbol{x}\|_p + \|\boldsymbol{y}\|_p$$

****

$$\sum_{j=1}^{\infty} |\xi_j \eta_j| \leqq \left(\sum_{k=1}^{\infty} |\xi_k|^p\right)^{1/p} \left(\sum_{m=1}^{\infty} |\eta_m|^q\right)^{1/q}$$

………………………………………….(4a)

where $p>1$ and $1/p+1/q=1$. This inequality was given by O. Hölder

Cauchy – Schwarz Inequality:

If $p=2$, then $q=2$

Equation (4a) gives,

for sums

$$\sum_{j=1}^{\infty} |\xi_j \eta_j| \leqq \sqrt{\sum_{k=1}^{\infty} |\xi_k|^2} \sqrt{\sum_{m=1}^{\infty} |\eta_m|^2}.$$

It is too early to say much about this case $p=q=2$ in which p equals its conjugate q, but we want to make at least the brief remark that this case will play a particular role in some of our later chapters and lead to a space (a Hilbert space) which is "nicer" than spaces with $p \neq 2$.

2.5.1 SOLVED PROBLEMS

Problem 1: Let $\mathbb{C}^n$ is a vector space over $\mathbb{C}$. Let p be a real number such that $1 \leq p < \infty$ for $x = (x_i)_{i=1}^n \in \mathbb{C}^n$, define

$$\|\boldsymbol{x}\|_p = [\sum_{i=1}^n |x_i|^p]^{1/p},$$

show that $\| \quad \|_p$ is a norm on $\mathbb{C}^n$.

Solution:

(i) $\forall\, x = (x_i)_{i=1}^n \in \mathbb{C}^n, |x_i| \geq 0 \forall i, 1 \leq i \leq n.$

Then, $\|\boldsymbol{x}\|_p = [\sum_{i=1}^n |x_i|^p]^{1/p} \geq 0.$

(ii) $\forall \alpha \in \mathbb{C}$,

$$\|\alpha \boldsymbol{x}\|_p = \left[\sum_{i=1}^n |\alpha x_i|^p\right]^{1/p}$$

$$=[\sum_{i=1}^n |\alpha|^p |x_i|^p]^{1/p}$$

$$= |\alpha| \left[\sum_{i=1}^n |x_i|^p\right]^{1/p}$$

$$= |\alpha| \|\boldsymbol{x}\|_p.$$

(iii) $\forall\; x = (x_i)_{i=1}^n \in \mathbb{C}^n, y = (y_i)_{i=1}^n \in \mathbb{C}^n,$

Then, $\|\boldsymbol{x} + \boldsymbol{y}\|_p \leq \|\boldsymbol{x}\|_p + \|\boldsymbol{y}\|_p$.[By Minkowski's inequality]

(iv) $\|\boldsymbol{x}\|_p = 0 \Leftrightarrow [\sum_{i=1}^n |x_i|^p]^{1/p} = 0.$

$$\Leftrightarrow \sum_{i=1}^n |x_i|^p = 0.$$

$$\Leftrightarrow x_i = 0 \forall i, 1 \leq i \leq n.$$

$$\Leftrightarrow x = (x_1, x_2, \ldots \ldots x_n) = (0,0, \ldots \ldots .0)$$

$$\Leftrightarrow x = 0.$$

Therefore, $(\mathbb{C}^n, \|\ \ \|_p)$ is a normed linear space denoted by $L_n^p, (1 \le p < \infty)$

Problem 2: Let $\mathbb{C}^n$ is a vector space over $\mathbb{C}$. for $x = (x_i)_{i=1}^n \in \mathbb{C}^n$, define

$$\|x\|_\infty = \max\{|x_1|, |x_2|, , \dots |x_n|\} = max_{1 \le i \le n}|x_i|,$$

show that $\|\ \ \|_p$ is a norm on $\mathbb{C}^n$.

Solution:

(i) $\forall\, x = (x_i)_{i=1}^n \in \mathbb{C}^n, |x_i| \ge 0 \forall i, 1 \le i \le n.$

Then, $\|x\|_\infty = sup_{1 \le i \le n}|x_i| \ge 0.$

(ii) $\forall \alpha \in \mathbb{C}, \forall\, x \in \mathbb{C}^n$

$$\|\alpha \boldsymbol{x}\|_\infty = sup_{1 \le i \le n}|\alpha x_i|$$

$$= |\alpha| sup_{1 \le i \le n}|x_i|$$

$$= |\alpha| \|\boldsymbol{x}\|_\infty.$$

(iii) $\forall\ x = (x_i)_{i=1}^n \in \mathbb{C}^n, y = (y_i)_{i=1}^n \in \mathbb{C}^n,$

$$\|x + y\|_\infty = sup_{1 \le i \le n}\{|x_1 + y_1|, |x_2 + y_2|,. \\ |x_i + y_i|, \dots |x_n + y_n|\}$$

$$\le sup_{1 \le i \le n}\{|x_1| + |y_1|, |x_2| + |y_2|,. \\ |x_i| + |y_i|, \dots |x_n| + |y_n|\}$$

$$\|x + y\|_\infty \le sup_{1 \le i \le n}|x_i| + sup_{1 \le i \le n}|y_i|$$

or, $\|x + y\|_\infty \le \|x\|_p + \|y\|_p$.[By definition of $\|\ \ \|_\infty$]

(iv) Now, $\|x\|_\infty = 0 \Leftrightarrow sup_{1 \le i \le n}|x_i|$

$$\Leftrightarrow sup_{1 \le i \le n}\{|x_1|, |x_2|, . |x_i|, \dots |x_n|\} = 0$$

$$\Leftrightarrow x_i = 0 \forall 1 \le i \le n$$

$$\Leftrightarrow x = (x_1, \dots . x_i \dots \dots x_n) = (0, \dots . .0, , , , 0) = 0$$

$(\mathbb{C}^n, \|\ \ \|_\infty)$ is a normed linear space, denoted L_n^∞.

Problem 3: Let $\mathbb{C}^n$ is a vector space over $\mathbb{C}$. for $x = (x_i)_{i=1}^{n} \in \mathbb{C}^n$, define $\|x\| = \sum_{i=1}^{\infty}|x_i| = |x_1| + |x_2| + \cdots . |x_n| + \ldots\ldots$

Check that $\| \quad \|$ is a norm on $\mathbb{C}^n$ or not.

Solution. $x = (1,1 \ldots .1 \ldots) \in \mathbb{C}^n = (1)_{n=1}^{\infty} = \infty$

$$\sum_{n=1}^{\infty}(\mathbf{1}) = \infty$$

$\| \quad \|$ is not a norm on $\mathbb{C}^n$.

2.6 HOLDER'S INEQUALITY FOR INFINITE SEQUENCES

Let $x = (x_i)_{i=1}^{\infty} \in L^p, (1 < p < \infty), y = (y_i)_{i=1}^{\infty} \in L^q$, q is the conjugate index of p, then

$$(x_n y_n)_{n=1}^{\infty} \in l^1$$

So $\sum_{n=1}^{\infty}|x_n y_n| \leq \left[\sum_{n=1}^{\infty}|x_n|^p\right]^{1/p}\left[\sum_{n=1}^{\infty}|y_n|^q\right]^{1/q}$.

…………………………………………. [5]

(first prove the finite part and then continue).

Proof:- Let m be any $+ve$ integer.

$\sum_{n=1}^{m}|x_n y_n| \leq \left[\sum_{n=1}^{m}|x_n|^p\right]^{1/p}\left[\sum_{n=1}^{m}|y_n|^q\right]^{1/q}$.

$\leq \left[\sum_{n=1}^{\infty}|x_n|^p\right]^{1/p}\left[\sum_{n=1}^{\infty}|y_n|^q\right]^{1/q} < \infty$ **…………[6]**

[as product of two finites is finite].

The m'th Partial sum of the series $\sum_{n=1}^{\infty}|x_n y_n|$ is odd $\forall m \geq 1$.

Taking $m \longrightarrow \infty$ in [6], we have,

$\sum_{n=1}^{\infty}|x_n y_n| \leq \left[\sum_{n=1}^{\infty}|x_n|^p\right]^{1/p}\left[\sum_{n=1}^{\infty}|y_n|^q\right]^{1/q}$.

Or, $\sum_{n=1}^{\infty}|x_n y_n| \leq \|x\|_p \, \|y\|_q$.

Remark:-

Let $x = (x_i)_{n=1}^{\infty} \in l^1$ and $y = (y_n)_{n=1}^{\infty} \in l^{\infty}$,

then

$(x_n y_n)_{n=1}^{\infty} \in l^1$ and $\sum_{n=1}^{\infty} |x_n y_n| \le \|x\|_p \; \|y\|_{\infty}$.

Proof:- $\|x\|_1 = \sum_{n=1}^{\infty} |x_n| < \infty$.

$\|y\|_{\infty} = max\,|y_n| < \infty$.

$$\sum_{n=1}^{\infty} |x_n y_n| = \sum_{n=1}^{\infty} |x_n||y_n| \le \{max_{n\ge 1}|y_n|\} \sum_{n=1}^{\infty} |x_n| = \|y\|_{\infty} \; \|x\|_1$$

$$\left(\because max_{n\ge 1}|y_n| = \; \|y\|_{\infty}; \sum_{n=1}^{\infty} |x_n| = \|x\| \right)$$

Or, $\sum_{n=1}^{\infty} |x_n y_n| \le \|x\|_p \; \|y\|_{\infty}$.

2.7 MINKOWSKI'S INEQUALITY FOR INFINITE SEQUENCES

Let $x = (x_i)_{n=1}^{\infty} \in l^P$ and $y = (y_n)_{n=1}^{\infty} \in l^P$, $((1 \le p < \infty)$,

Define,

$\|\boldsymbol{x}\|_p = [\sum_{n=1}^{\infty} |x_n|^p]^{1/p}$.

Then, $\|\boldsymbol{x} + \boldsymbol{y}\|_p \le \|\boldsymbol{x}\|_p + \|\boldsymbol{y}\|_p$.

………………..[7]

Proof.

$$\sum_{i=1}^{n} |x_i + y_i|^p \le \sum_{i=1}^{n} |x_i| + |y_i| \, |x_i + y_i|^{p-1}$$

$$\le \sum_{i=1}^{n} |x_i| \, |x_i + y_i|^{p-1} + \sum_{i=1}^{n} |y_i| \, |x_i + y_i|^{p-1}$$

$$\le \left[\sum_{i=1}^{n} |x_i|^p\right]^{1/p} \left[\sum_{i=1}^{n} (|x_i + y_i|^{p-1})^q\right]^{1/q}$$

$$+\left[\sum_{i=1}^{n} |y_i|^p\right]^{1/p} \left[\sum_{i=1}^{n} (|x_i + y_i|^{p-1})^q\right]^{1/p}$$

(By Holder's inequality for infinite sequence)

$$\sum_{i=1}^{n} |x_i + y_i|^p = \|\boldsymbol{x}\|_p \|x + y\|_p^{p/q} + \|\boldsymbol{y}\|_p \|x + y\|_p^{p/q}$$

i.e. $\sum_{i=1}^{n} |x_i + y_i|^p \le \left(\|\boldsymbol{x}\|_p + \|\boldsymbol{y}\|_p\right) \|x + y\|_p^{p/q}$.

Therefore taking the $limit_{n\to\infty}$, we have,

or, $\|x + y\|_p^p \le \left(\|\boldsymbol{x}\|_p + \|\boldsymbol{y}\|_p\right) \|x + y\|_p^{p/q}$.

or, $\|x + y\|_p^{p-p/q} \le \|\boldsymbol{x}\|_p + \|\boldsymbol{y}\|_p$

or, $\|\boldsymbol{x} + \boldsymbol{y}\|_p \le \|\boldsymbol{x}\|_p + \|\boldsymbol{y}\|_p$.

Alternate proof:

(first prove the finite part and then continue).

Let any integer $m \ge 0$.

$[\sum_{i=1}^{n} |x_i + y_i|^p]^{1/p} \le [\sum_{i=1}^{n} |x_i|^p]^{1/p} + [\sum_{i=1}^{n} |y_i|^p]^{1/p}$ **…..[8]**

(by Minkowskian inequality for finite sequence)

Therefore, $[\sum_{i=1}^{n} |x_i + y_i|^p]^{1/p} \le [\sum_{i=1}^{\infty} |x_i|^p]^{1/p} + [\sum_{i=1}^{\infty} |y_i|^p]^{1/p}$.

Taking limit as $n \to \infty$, we get,

$$\left[\sum_{i=1}^{n} |x_i + y_i|^p\right]^{1/p} \le \|x\|_p + \|y\|_p$$

or, $$\|x + y\|_p \le \|x\|_p + \|y\|_p.$$

Remark:

Let $x = (x_i)_{n=1}^{\infty} \in l^P$ and $y = (y_n)_{n=1}^{\infty} \in l^P$, $((1 \le p < \infty)$,

Define,

$\|\boldsymbol{x}\|_p = max_{n\geq 1}\{|x_n + y_n|\}$.

$\leq max_{n\geq 1}\{|x_n| + |y_n|\}$,(Since $|x_n + y_n| \leq |x_n| + |y_n|$)

$\leq max_{n\geq 1}\{|x_n|\} + max_{n\geq 1}\{|y_n|\}$

$$= \|x\|_\infty + \|y\|_\infty$$

Then, $\|\boldsymbol{x}+\boldsymbol{y}\|_\infty \leq \|\boldsymbol{x}\|_\infty + \|\boldsymbol{y}\|_\infty$.

………………..[9]

Example:

$$x = (x_i)_{n=1}^{\infty} = \left(\frac{1}{n^{2/3}}\right)_{n=1}^{\infty}; p = 3, q = \frac{3}{2},$$

$\frac{1}{3} + \frac{1}{3/2} = 1;$

$\frac{1}{p} + \frac{1}{q} = 1.$

$\sum_{n=1}^{\infty} |\boldsymbol{x_n}|^{\boldsymbol{p}} = \sum_{n=1}^{\infty} \left(\frac{1}{n^{2/3}}\right)^{\mathbf{3}} = \sum_{n=1}^{\infty} \frac{\mathbf{1}}{n^2} < \infty.$

& $\sum_{n=1}^{\infty} |\boldsymbol{x_n}|^{\boldsymbol{q}} = \sum_{n=1}^{\infty} \left(\frac{1}{n^{2/3}}\right)^{\mathbf{3/2}} = \sum_{n=1}^{\infty} \frac{\mathbf{1}}{n} = \infty$

Therefore, $x \in l^3 \Rightarrow x \in l^p$ and $x \notin l^{3/2} \Rightarrow x \notin l^q$.

So, $x \in l^p$ it does not implies $y \notin l^q$. The two different $x \in l^p, y \in l^q$ must be assumed simultaneously when we consider Holder's inequality.

Example:

$$\boldsymbol{x} = (\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots\ldots \boldsymbol{x_i} \ldots\ldots \boldsymbol{x_n}) \in \mathbb{C}^{\boldsymbol{n}}.$$

$\|\boldsymbol{x}\|_p = [\sum_{n=1}^{\infty} |x_n|^p]^{1/p}, 1 \leq p < \infty.$

$\|\boldsymbol{x}\|_\infty = Max_{1\leq i\leq n}\{|x_i|\}.$

$$x \in l_n^p, 1 \leq p < \infty.$$

Then $\|\boldsymbol{x}\|_\infty = limit_{p\to\infty}\|x\|_p$.

i.e. $\max\{|x_1|, |x_2|, \ldots\ldots, |x_i|, \ldots\ldots |x_n|\} = limit_{\,p\to\infty}[\sum_{n=1}^{\infty} |x_n|^p]^{1/p}$.

Proof. Suppose $n = 2, x_1, x_2$ are $+ve$ real numbers.

$\|\boldsymbol{x}\|_\infty = \max\{\boldsymbol{x_1}, \boldsymbol{x_2}\} \leq \left[x_1^p + x_2^p\right]^{1/p} = \|x\|_p$.

Let $x_1 = x_2$. In this case,

$$limit_{p\to\infty}\|x\|_p = limit_{p\to\infty}\left[x_1^p + x_2^p\right]^{1/p} = limit_{p\to\infty}\left[x_2^p + x_2^p\right]^{1/p}$$

$$= limit_{p\to\infty}\left[2x_2^p\right]^{1/p} = limit_{p\to\infty}\left[2^{1/p}x_2\right] = 2^0\, x_2 = x_2$$

$$= \|x\|_\infty$$

$limit_{p\to\infty}\|x\|_p = \|x\|_\infty$

Let $0 < x_1 < x_2$.

In this case,

$$limit_{p\to\infty}\|x\|_p = limit_{p\to\infty}\left[x_1^p + x_2^p\right]^{1/p}$$

$$= limit_{p\to\infty}\left[\left(\frac{x_1}{x_2}\right)^p + 1\right]^{1/p}. x_2 = 1. x_2, as\ \frac{x_1}{x_2} < 1$$

$= \max\{\boldsymbol{x_1}, \boldsymbol{x_2}\}$

Therefore

$limit_{p\to\infty}\|x\|_p = \|x\|_\infty$.

2.7.1 SOLVED PROBLEMS

Problem 1:

Let $\mathcal{R}[0,1]$ be a set of all Riemann – integrable functions over $[0,1]$,

i.e. $f \in \mathbb{R}[0,1] \Rightarrow \int_0^1 |f(x)|\, dx < \infty$.

Define $\|f\| = \int_0^1 |f(x)| dx$

Verify $\|\quad\|$ is a norm on $\mathcal{R}[0,1]$ or not.

Proof:-

i. $\forall f \in \mathbb{R}[0,1], \int_0^1 |f(x)|\, dx \geq 0 \Longrightarrow \|f\| \geq 0.$

ii. Let α be a real number,

$\|\alpha f\| = \int_0^1 |(\alpha f)(x)|\, dx = |\alpha| \int_0^1 |f(x)| dx = |\alpha| \|f\|$

iii. $\forall f, g \in \mathbb{R}[0,1]$

$$\|f + g\| = \int_0^1 |f(x) + g(x)| dx \leq \int_0^1 |f(x)| dx + \int_0^1 |g(x)|\, dx$$

$$\leq \|f\| + \|g\|$$

iv. Define $f(x) = \begin{cases} 1, & x = 0 \\ 0, 0 < x \leq 1 \end{cases}$

$$f \neq 0 \forall x \in [0,1]$$

But $\|f\| = \int_0^1 |f(x)| dx = 0$ so $f \neq 0, \|f\| = 0.$

Therefore $\| \quad \|$ is not a norm but a semi-norm.

Problem 2:

Let $x, y \in X$ where $(X, \| \quad \|)$ is a normed linear space. Then,

$$|\|x\| - \|y\|| \leq \|x - y\| \forall x, y \in X.$$

……………………………………………..second triangle inequality.

Proof. $\forall x, y \in X$, Let $x = (x - y) + y$. $\|x\| = \|(x - y) + y\|$

$\leq \|x - y\| + \|y\|$, by Triangle's Inequality

$$\|x\| - \|y\| \leq \|x - y\| \forall x, y \in X \ldots \ldots \ldots \ldots (1)$$

As this is true for all x and y, interchanging x and y in (1), we have

$$\|y\| - \|x\| \leq \|y - x\| \forall x, y \in X$$

Or $-(\|x\| - \|y\|) \leq \|(-1)(x - y)\|$

$= (-1)\|x - y\| = \|x - y\|$…………..(2)

By from (1) and (2),

$$|\|x\| - \|y\|| \leq \|x - y\| \forall x, y \in X.$$

Problem 3:

Let $(X, \|\quad\|)$ and $(Y, \|\quad\|)$ be two normed linear space over K($\mathbb{R}$ or $\mathbb{C}$). Then $X \times Y$ is normed linear space.

The norm on $X \times Y$ can be defined in one of following ways.

a. $\|(x,y)\|_1 = \|x\| + \|y\|, \forall x, y \in X$.

b. $\|(x,y)\|_p = (\|x\|^p + \|y\|^p)^{1/p}, 1 \le p < \infty$.

Solutions:-

a.

i. $\forall x \in X, \|x\| \ge 0$

$\forall y \in Y, \|y\| \ge 0$

Therefore, $\|(x,y)\|_1 = \|x\| + \|y\| \ge 0$.

Therefore, , $\|(x,y)\|_1 \ge 0, \forall x, y \in X \times Y$.

ii. $\forall \alpha \in K$,

Therefore, , $\|\alpha(x,y)\|_1 = \|(\alpha x, \alpha y)\|_1$

$= \|\alpha x\| + \|\alpha y\|$

$= |\alpha|\|x\| + |\alpha|\|y\|$

$= |\alpha|(\|x\| + \|y\|)$

$= |\alpha|\|(x,y)\|_1$

iii. $\|(x_1, y_1) + (x_2, y_2)\|_1 = \|(x_1 + x_2, y_1 + y_2)\|_1$

$= \quad \|x_1 + x_2\| + \|y_1 + y_2\|$

$\le (\|x_1\| + \|x_2\|) + (\|y_1\| + \|y_2\|)$[by triangle inequality]

$=(\|x_1\| + \|y_1\|) + (\|x_2\| + \|y_2\|)$

$\|(x_1, y_1) + (x_2, y_2)\|_1 \le \|(x_1, y_1)\|_1 + \|(x_2, y_2)\|_1$.

iv. $\|(x,y)\|_1 = 0 \Leftrightarrow \|x\| + \|y\|=0$

$\Leftrightarrow \|x\|= 0$ and $\|y\|=0$

$\Leftrightarrow x= 0$ and $y = 0$

$\Leftrightarrow (x,y) = (0,0) = 0$

It means $(X \times Y, \|\quad\|_1)$ is a normed linear space.

b.

i. $\forall x \in X,\ \forall y \in Y, \|x\| \geq 0, \|y\| \geq 0$

$\|\boldsymbol{x}\|^{\boldsymbol{p}} \geq 0, \|\boldsymbol{y}\|^{\boldsymbol{p}} \geq 0, 1 \leq p < \infty$. Therefore,

$\|(x,y)\|_p = \{\|x\|^p + \|y\|^p\}^{1/p} \rightarrow 0.$

ii. $\forall \alpha \in K, \forall (x,y) \in X \times Y,$

$$, \|\alpha(x,y)\|_p = \|(\alpha x, \alpha y)\|_p = \{\|\alpha x\|^p + \|\alpha y\|^p\}^{1/p}$$

$$= \{(|\alpha|\|x\|)^p + (|\alpha|\|y\|)^p\}^{1/p} = |\alpha|[\|x\|^p + \|y\|^p]^{1/p}$$

$$= |\alpha|\|(x,y)\|_p.$$

iii. $(x_1, y_1), (x_2, y_2) \in X \times Y$.

$$\|(x_1,y_1) + (x_2,y_2)\|_p = \|(x_1 + x_2, y_1 + y_2)\|_p$$

$$\leq \{[\|x_1\| + \|x_2\|]^p + [\|y_1\| + \|y_2\|]^p\}^{1/p}.$$

$$\|(\|x_1\|, \|y_1\|) + (\|x_2\|, \|y_2\|)\|_p$$

$$\|(x_1,y_1) + (x_2,y_2)\|_p$$

$$\leq \|(\|x_1\|, \|y_1\|)\|_p + \|(\|x_2\|, \|y_2\|)\|_p$$

(By Minkowski's Inequality in $\mathbb{C}^2$)

$$= \|(x_1,y_1)\|_p + \|(x_2,y_2)\|_p$$

Since $(\|x_1\|, \|y_1\|)_p = \{\|x_1\|^p + \|y\|^p\}^{1/p} = \|(x_1,y_1)\|_p$

$$\|(x_1,y_1) + (x_2,y_2)\|_p \leq \|(x_1,y_1)\|_p + \|(x_2,y_2)\|_p.$$

iv. Therefore, $\|(x,y)\|_p = 0 \Leftrightarrow \{\|x\|^p + \|y\|^p\}^{1/p} = 0,$

$1 \leq p < \infty.$

$$\Leftrightarrow \|x\|^p + \|y\|^p = 0$$

$$\Leftrightarrow \|x\|^p = 0, \|y\|^p = 0$$

$$\Leftrightarrow \|x\| = 0, \|y\| = 0$$

$$\Leftrightarrow x = 0, y = 0$$

$$\Leftrightarrow (x,y) = (0,0) = 0$$

$$\Leftrightarrow (x,y) = 0.$$

It means $(X \times Y, \|\ \ \|_p)$ is a normed linear space.

2.8 SUMMARY

Present unit is presentation of the topic Extended Real Number System, Holder's Inequality for finite sequence, Minkowski's Inequality for finite sequences and then Solved Problems discussed on above mentioned topic. Then Holder's Inequality for infinite sequence, Minkowski's Inequality for infinite sequences and Solved Problems discussed here.

2.9 GLOSSARY

i. **Set:** Any well-defined collection of objects or numbers are referred to as a set.

ii. **Interval:** An open interval does not contain its endpoints, and is indicated with parentheses. $(a, b) =]a, b[= \{x \epsilon \mathbb{R}: a < x < b\}$. A closed interval is an interval which contain all its limit points, and is expressed with square brackets. $[a, b] = [a, b] = \{x \epsilon \mathbb{R}: a \leq x \leq b\}$. A half-open interval includes only one of its endpoints, and is expressed by mixing the notations for open and closed intervals. $(a, b] =]a, b] = \{x \epsilon \mathbb{R}: a < x \leq b\}$. $[a, b) = [a, b[= \{x \epsilon \mathbb{R}: a \leq x < b\}$.

iii. **Ordered Pairs:** An ordered pair (a, b) is a set of two elements for which the order of the elements is of significance. Thus $(a,b) \neq (b,a)$ unless $a = b$. In this respect (a, b) differs from the set $\{a,\ b\}$.Again $(a,b) = (c,d) \Leftrightarrow a = c$ and $b = d$ .If X and Y are two sets, then the set of all ordered pairs (x, y), such that $x \in X$ and $y \in Y$ is called Cartesian product of X and Y.

iv. **Relation:** A subset R of $X \times Y$ is called relation of X on Y. It gives a correspondence between the elements of X and Y. If (x, y) be an element of R, then y is called image of x. A relation in which each element of X has a single image is called a function.

v. **Function:** Let X and Y are two sets and suppose that to each element x of X corresponds, by some rule, a single element y of Y. Then the set of all ordered pairs (x, y) is called function.

vi. **Variable:** A symbol such as x or y, used to represent an arbitrary element of a set is called a variable.

vii. **Metric space:** Let $X \neq \emptyset$ be a set then the metric on the set X is defined as a function $d: X \times X \to [0, \infty)$ such that some conditions are satisfied.

viii. **Vector space: -** Let V be a nonempty set with two operations

(i) **Vector addition:** If any $u, v \in V$ then $u + v \in V$

(ii) **Scalar Multiplication:** If any $u \in V$ and $k \in F$ then $ku \in V$

Then V is called a vector space (over the field F) if the following axioms hold for any vectors if the some conditions hold.

CHECK YOUR PROGRESS

1. Young's inequality is ……………………………

2. Holder's Inequality For Finite Sequences is……………………

3. Minkowski's Inequality For Finite Sequences…………………

4. Holder's Inequality For Infinite Sequences……………………

5. Minkowski's Inequality For Infinite Sequences………………

2.10 REFERENCES

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2.11 SUGGESTED READINGS

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2.12 TERMINAL QUESTIONS

1. Let $(X, \|\quad\|)$ and $(Y, \|\quad\|)$ be two normed linear space over $K(\mathbb{R}$ or $\mathbb{C})$. $\|(x,y)\|_\infty = \max\{\|x\|, \|y\|\} \forall x, y \in X$. Then $(X \times Y, \|\quad\|_{\infty.})$ is normed linear space.

2. Let $(X_1, \|.\|_1)$ and $(X_2, \|.\|_2)$ be n normed linear space and $X = X_1 \times X_2 \times .. \times X_n$ show that,

a. $\|(x_1, x_2, \ldots\ldots, x_n)\|_1 = \|x_1\|_1 + \|x_2\|_1 + \cdots .. \|x_n\|_n,$

$$\forall (x_1, x_2, \ldots\ldots, x_n) \in X_1 \times X_2 \times .. \times X_n \,.$$

b. $\|(x_1, x_2, \ldots\ldots, x_n)\|_\infty = \max\{\|x_1\|_1, \|x_2\|_1, \ldots . \|x_n\|_n\}$

c. $\|(x_1, x_2, \ldots\ldots, x_n)\|_p = \left\{\|x_1\|_1{}^p + \|x_2\|_2{}^p + \cdots \|x_n\|_n{}^p\right\}^{1/p}$

are norms on $X = X_1 \times X_2 \times .. \times X_n$.

3. Let l^p be the $p -$ summable sequence of complex numbers $((1 \le p < \infty)$ for $x \in l^p$, define $\|\quad\|_p = \left[\sum_{n=1}^{\infty} |x_n|^p\right]^{1/p}$. Verify that $\|\quad\|_p$ is a norm.

4. Let l^∞ be the vector space of all complex valued odd sequence for $x \in l^\infty$, define $\|\quad\|_p = sup_{n \ge 1} |x_n|$ Verify that $\|\quad\|_\infty$ is a norm.

2.13 ANSWERS

CHECK YOUR PROGRESS

1. $ab \le \frac{a^p}{p} + \frac{b^q}{q}$
2. $\sum_{i=1}^{n} |x_i y_i| \le \|x\|_p \|y\|_q$.
3. $\|\boldsymbol{x} + \boldsymbol{y}\|_p \le \|\boldsymbol{x}\|_p + \|\boldsymbol{y}\|_p$
4. $\sum_{n=1}^{\infty} |x_n y_n| \le \|x\|_p \, \|y\|_q$
5. $\|\boldsymbol{x} + \boldsymbol{y}\|_p \le \|\boldsymbol{x}\|_p + \|\boldsymbol{y}\|_p$.

UNIT 3:

BANACH SPACE

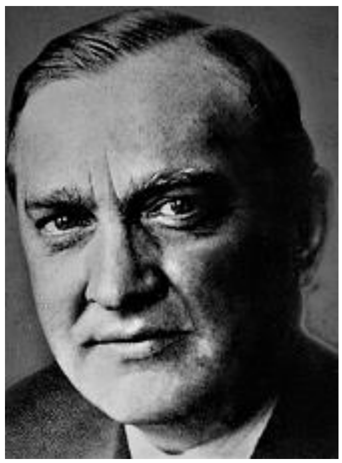
CONTENTS:

3.1 INTRODUCTION

Before this unit we are completely familiar with normed space. In present unit we are explaining about Banach space. Now In continuation Banach space is a vector space with a metric that allows the computation of vector length and distance between vectors and is complete in the sense that a Cauchy sequence of vectors always converges to a well-defined limit that is within the space.

Polish mathematician Stefan Banach, who introduced the concept of Banach space and studied it systematically in **1920–1922**. Discovery of this concept Hans Hahn and Eduard Helly also helped to Stefan Banach.

Maurice René Fréchet was the first to use the term "Banach space" and Banach in turn then coined the term "Fréchet space" Banach spaces for the function spaces studied by Hilbert, Fréchet, and Riesz earlier in the century. Banach spaces play a main role in functional analysis. In other areas of analysis, the spaces under study are often Banach spaces.



Ref: https://en.wikipedia.org/

Fig 3.1

(Stefan Banach 30 March 1892 – 31 August 1945)

3.2 OBJECTIVES

After studying this unit, learner will be able to

i. Defined the concept of *Banach space*

ii. Describe the concept of completeness.

iii. Problems, Theorems and examples related to *normed space* and *Banach space*.

3.3 CONTINUOUS AT A POINT

Continuity at a point:

Let (X, d) and (Y, d^*) be two metric spaces and let $f: X \to Y$ be a function of X into Y. F is continuous at a point $l \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d^*(f(x), f(l)) < \varepsilon \quad \text{whenever } 0 < d(x, l) < \delta.$$

Continuous map:

A function f of a metric space (X, d) into another metric space (Y, d^*) is said to be continuous if it is continuous at every point of X.

Proposition:

Norm is a continuous function in a normed linear space.

Proof: Let $(X, \| \quad \|)$ be a normed linear space.

Define $\theta: X \to \mathbb{R}$ be a function defined by $\theta(x) = \|x\| \; \forall x \in X$.

Let$d(x, y) = \|x - y\| \forall x, y \in X$.

$\rho(\alpha, \beta) = |\alpha - \beta| \forall \alpha, \beta \in \mathbb{R}$.

Now, $x, x_0 \in X$,

$\rho(\theta(x), \theta(x_0)) = \rho(\|x\|, \|x_0\|) = |\|x\| - \|x_0\|| \leq \|x - x_0\|$,

by second triangle inequality,

$\rho(\theta(x), \theta(x_0)) = d(x, x_0) < \in$, whenever $d(x, x_0) < \delta = \in$,

or, $\rho(\theta(x), \theta(x_0)) < \in$, whenever $d(x, x_0) < \delta = \in$,

Norm is continuous at $x_0 \in X$. Since x_0 is an arbitrary member of X, therefore norm is continuous on X.

Proposition:

Addition is a continuous function in a normed linear space.

Proof. Let $(X, \| \quad \|)$ be a normed linear space.

Define $\theta: X \times X \rightarrow X$ be a function defined by

$\theta(x, y) = x + y \; \forall x, y \in X \times X.$

i.e.

$$\theta: (X \times X, d) \rightarrow (X, \rho).$$

Let $d\big((x, y), (x', \; y')\big) = \|(x, y) - (x', y')\|_1$

$$= \|x - x'\| + \|y - y'\| < \in$$

$$(as \; \|x, y\|_1 = \|x\|_1 + \|y\|_1).$$

Let $\rho\big(\theta(x, y), \; \theta(x_0, \; y_0)\big) = \|x - y\|,$ so for all $(x_0, y_0) \in X \times X,$

$$\rho(\theta(x, y), \theta(x_0, y_0)) = \rho(x + y, x_0 + y_0) = \|(x + y) - (x_0 + y_0)\|$$

$$= \|(x - x_0) + (y - y_0)\|$$

By triangle inequality of norm

$$= \|(x, y)_1 - (x_0, y_0)\| = d\big((x, y), (x_0, y_0)\big) < \in,$$

(whenever $d\big((x, y), (x_0, y_0)\big) < \in$)

It proves that the sum functions is continuous in a normed linear space.

Remark:

Let $(X, \| \quad \|)$ be a normed linear space be a normed linear space over

$\mathbb{K} = (\mathbb{R} \; or \; \mathbb{C}).$

$$\mathbb{K} \times X = \{(\alpha, x): \alpha \in \mathbb{K}, x \in X\}.$$

$\mathbb{K} \times X$ is a vector space under following operations over $\mathbb{K}$.

- **i.** $(\alpha, x) + (\beta, y) = (\alpha + \beta, x + y)$
- **ii.** $\beta(\alpha, x) = (\beta\alpha, \beta x).$

Define a norm $\| \quad \|_1$ on $\mathbb{K} \times X$ by $\|(\alpha, x)\|_1 = |\alpha| + \|x\|.$

i. $\forall\alpha \in \mathbb{K}$, $|\alpha| = 0; \forall x \in X, \|x\|\geq 0$.

$\|(\alpha, x)\|_1 \geq 0, \forall(\alpha, x)\epsilon\mathbb{K} \times X$.

ii. $\forall\beta \in \mathbb{K}$,

$$\|\beta(\alpha, x)\|_1 = \|(\beta\alpha, \beta x)\|_1 = |\beta\alpha| + \|\beta x\|$$
$$= |\beta||\alpha| + |\beta|\|x\|$$
$$= |\beta|[|\alpha| + \|x\|]$$
$$\|\beta(\alpha, x)\|_1 = |\beta|\|(\alpha, x)\|_1$$

iii. $\|(\alpha, x) + (\gamma, y)\|_1 = \|\alpha + \gamma, x + y\| = |\alpha + \gamma| + \|x + y\|$

$$\leq |\alpha| + |\gamma| + \|x\| + \|y\|$$
$$= (|\alpha| + \|x\|) + (|\gamma| + \|y\|) = \|(\alpha, x)\|_1 + \|(\gamma, y)\|_1$$
$$\|(\alpha, x) + (\gamma, y)\|_1 = \|(\alpha, x)\|_1 + \|(\gamma, y)\|_1$$

iv. $\|\alpha, x\|_1 = 0 \Leftrightarrow |\alpha| + \|x\| = 0$

$$\Leftrightarrow |\alpha| = 0, \|x\| = 0$$
$$\Leftrightarrow \alpha = 0, x = 0$$
$$\Leftrightarrow (\alpha, x) = (0,0) = 0$$
$$\Leftrightarrow (\alpha, x) = (0,0) = 0$$
$$\Leftrightarrow (\alpha, x) = 0$$

$(\mathbb{K} \times X, \| \quad \|_1)$ is a normed linear space.

Note: $d\big((\alpha_n, x_n), (\alpha_0, x_0)\big) = \|(\alpha_n, x_n) - (\alpha_0, x_0)\|$

$= \|(\alpha_n - x_n, x_n - x_0)\|_1 = |\alpha_n - x_n| + \|x_n - x_0\|$.

Definition of convergence:

From above, it follows that, $((\alpha_n, x_n))_{n=1}^{\infty}$ in $\mathbb{K} \times X$ converges to (α_0, x_0) in $\mathbb{K} \times X$ iff

$|\alpha_n - x_n| \to 0$ as $n \longrightarrow \infty$ iff $(\alpha_n)_{n=1}^{\infty}$ converges to x_0 in X.

Or

$x_n \to x$ if $\|x_n - x_0\| \to 0$ as $n \longrightarrow \infty$

Proposition:-

The scalar multiplication is continuous in a normed linear space.

Proof:

Let $(X, \| \quad \|)$ be a normed linear space be a normed linear space over $\mathbb{K} = (\mathbb{R}\ or\ \mathbb{C})$.

Then, $(\mathbb{K} \times X, \| \quad \|_1)$ is a normed linear space.

Refine θ by $\theta(\alpha, x) = \alpha . x\ \forall\ (\alpha, x) \in \mathbb{K} \times X$.

Let $((\alpha_n, x_n))_{n=1}^{\infty}$ be a convergent sequence in $\mathbb{K} \times X$, and this sequence $((\alpha_n, x_n))_{n=1}^{\infty}$ converges to (α, x) in $\mathbb{K} \times X$.

We need to show that the sequence $(\theta(\alpha_n, x_n))_{n=1}^{\infty}$ converges to $\theta(\alpha, x)$,

For this,

$$d\big(\theta(\alpha_n, x_n), \theta(\alpha, x)\big) = \|\theta(\alpha_n, x_n) - \theta(\alpha, x)\|_1 = \|\alpha_n, x_n - \alpha x\|_1$$

$$\big(as\ \theta\ (\alpha, x)\big) = \alpha, x\ \forall\ (\alpha, x) \in \mathbb{K} \times X)$$

$$= \|(\alpha_n - \alpha)x_n + \alpha(x_n - x)\|_1 \leq \|(\alpha_n - \alpha)x_n\|_1 + \|\alpha(x_n - x)\|_1$$

$$= (|\alpha_n - \alpha|\|x_n\|_1) + (|\alpha|\|x_n - x\|_1) \ldots \ldots \ldots \ldots (1)$$

Since $(x_n)_{n=1}^{\infty}$ in x in X.

Therefore $(x_n)_{n=1}^{\infty}$ is bounded.

There exists a real number $M \geq 0$ such $\|x_n\| \leq M\ \forall\ n \geq 1$.

Then by (1),

$$d\big(\theta(\alpha_n, x_n), \theta(\alpha, x)\big) \leq M|\alpha_n - \alpha|_1 + |\alpha|\|x_n - x\|_1 \ldots \ldots \ldots (2)$$

Again $(x_n)_{n=1}^{\infty}$ converges to x in X, then, for $\in > 0$,therefore, $\exists$ a positive integer $N_1 > 0$ such that

$$\|x_n - x\| < {}^{\varepsilon}/_{2(1 + |\alpha|)}, \forall\ n \geq N_1 \ldots \ldots (3)$$

Also, $(\alpha_n)_{n=1}^{\infty}$ in $\mathbb{K}$ converges to α in $\mathbb{K}$ for ${}^{\varepsilon}/_{2M} > 0$,

There exists a positive integer $N_2 > 0$ such that

$$|\alpha_n - \alpha| < {}^{\varepsilon}/_{2M}\ \forall\ n \geq N_2 \ldots \ldots \ldots \ldots (4)$$

Choose $N = \max\{N_1, N_2\}$.

By (2), (3) and (4), we have,

$$\|\theta(\alpha_n, x_n) - \theta(\alpha, x)\| \le M\frac{\varepsilon}{2M} + |\alpha|\frac{\varepsilon}{2(1+|\alpha|)} \forall\, n \ge N$$

$$< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2}.1 = \varepsilon \qquad \forall\, n \ge N.$$

Therefore ,

$\theta(\alpha_n, x_n)_{n=1}^{\infty}$ converges to $\theta(\alpha, x)$. Hence the function θ is continuous.

3.4CAUCHY SEQUENCE

Cauchy sequence:

Let d be a metric on a set X.

A sequence $\{x_n\}$ in the set X is said to be a *Cauchy sequence* if, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n, x_m) < \varepsilon \text{ whenever } n, m \ge n_0$$

Example

The sequence $\{x_n\}$ where$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, does not satisfy Cauchy's criterion of convergence. Indeed,

$$|x_{2n} - x_n| = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$

$$\le \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n}$$

$$= \frac{n}{2n} = \frac{1}{2}$$

So, $|x_n - x_m|$ *is not tends to* 0.

Theorem 1.

A convergent sequence in a metric space is a *Cauchy sequence*.

Proof.

Let $\{x_n\}$ be a sequence in a set X with metric d.

Let x be an element of X such that $\lim_{n\to\infty} x_n = x$.

Given any $\varepsilon > 0$, there exists some natural number m such that

$$d(x_n, x) < \frac{\varepsilon}{2} = 2 whenever\ n \geq m.$$

Assume any natural numbers n and n' such that $n \geq m$ and $n' \geq m$.

Then $d(x_n, x) < \frac{\varepsilon}{2}$ and $d(x_{n'}, x) < \frac{\varepsilon}{2}$.

Hence

$$d(x_n, x_{n'}) \leq d(x_n, x) + d(x_{n'}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Remark: The converse of the above result is not necessarily true.

Proof: Let $X = (0, 1]$.

Define a metric d on X by

$d(x, y) = |x - y| \forall x, y \in X$. (X, d) is metric space.

Consider the sequence $(\alpha_n)_{n=1}^{\infty} = \left(\frac{1}{n}\right)_{n=1}^{\infty}$.

For $0 < \in < 1$, choose a $+ve$ integer $N > \frac{2}{\in}$.

Mathematically, we choose, $N = \left[\frac{2}{\in}\right] + 1$.

Therefore $d(\alpha_m, \alpha_n) = |\alpha_m - \alpha_n|, m, n > N$

$$= \left|\frac{1}{m} - \frac{1}{n}\right|, m, n > N \leq \frac{1}{m} + \frac{1}{n}$$

$$< \frac{\in}{2} + \frac{\in}{2}, if\ , m, n > N \left(\because m > \frac{2}{\in}, n > \frac{2}{\in}\right) = \epsilon$$

$d(\alpha_m, \alpha_n) < \epsilon \forall m, n \geq N$.

and $(\alpha_n)_{n=1}^{\infty} = \left(\frac{1}{n}\right)_{n=1}^{\infty}$, is a Cauchy's sequence in X.

The sequence $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ does not converge in $X = (0, 1]$, because, 0 is not a member of X.

A sequence $\{x_n\}$ in a normed space X is said to be a *Cauchy sequence* if, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\|x_m - x_n\| < \varepsilon \text{ , } \forall n, m \geq n_0$$

If $S_n = x_1 + x_2 + \cdots + x_n$, where $n = 1,2,3 ...$ If (s_n) is convergent, say $s_n \to s$, that is $\|s_n - s\| \to 0$,

Then the infinite series or, briefly, series

$$\sum_{k=1}^{\infty} x_k = x_1 + x_2 + \cdots$$

is said to converge or to be *convergent*, s called the sum of the series and

$s = \sum_{k=1}^{\infty} x_k = x_1 + x_2 + \cdots$ (A)

If $\|x_1\| + \|x_2\| + \cdots$converges, the series (A) is said to be absolutely convergent.

If a normed space X contains a sequence (e_n) with the property that for every $x \in X$ there is a unique sequence of scalars (α_n) such that

$\|x - (\alpha_1 e_1 + \cdots + \alpha_n e_n)\| \to 0$ (as $n \longrightarrow \infty$).

Then (e_n) is called a Schauder basis (or basis) for X.

The series,

$\sum_{k=1}^{\infty} \alpha_k e_k$,

which has the sum x is then called the expansion of x with respect to (e_n), and we write,

$x = \sum_{k=1}^{\infty} \alpha_k e_k$,

---

l^p has a Schauder basis, namely (e_n), where $e_n = (\delta_{nj})$, that is, e_n is the sequence whose nth term 1 and all other terms are zero; thus

$$e_1 = (1,0,0,0..)$$

$$e_2 = (0,1,0,0..)$$

$$e_3 = (0,0,1,0..)$$

If a normed space X has a Schauder basis, then X is separable.

- Let $X = (X, \|.\|)$ be a normed space. Then there is a Banach space $\hat{X}$ and an isometry A from X onto a subspace W of $\hat{X}$ which is dense in $\hat{X}$. The space $\hat{X}$ is unique, expect for isometries.

3.5 COMPLETENESS

Complete metric space: A metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent.

Example:

$d(x, y) = |x - y|\ for\ x, y \in \mathbb{R}$; is complete metric space.

$d(z, w) = |z_1 - z_2| for\ z_1 - z_2 \in \mathbb{C}$ is complete metric space.

$$d(x, y) = \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{\frac{1}{2}}$$

and $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$ is complete metric space.

Subsequence: Let $\{x_n\}$ be a given sequence in a metric space (X, d) and let $\{n_k\}_{k\geq 1}$ be a sequence of positive integers such that $n_1 < n_2 < n_3 <..$ Then the sequence $\{x_{n_k}\}$ is called a subsequence of $\{x_n\}$.

Sub sequential limit: If $\{x_{n_k}\}$ converges, its limit is called a sub sequential limit of $\{x_n\}$.

NOTE: A sequence $\{x_n\}$ in X converges to x if and only if every subsequence of it converges to x.

Theorem2. If a Cauchy sequence of points in a metric space (X, d) contains a convergent subsequence, then the sequence converges to the same limit as the subsequence.

Proof. Let $\{x_n\}$ be a Cauchy sequence in (X, d).
Then for every positive number ε there exists an integer $m(\varepsilon)$ such that
$d(x_n, x_{n'}) < \varepsilon$ whenever $n, n' \geq m(\varepsilon)$
Let$\{x_{n_k}\}$be a convergent subsequence of $\{x_n\}$ and its limit by x.
It implies that $d(x_{n_{n'}}, x_n) < \varepsilon$ whenever $n, n' \geq m(\varepsilon)$
As $\{n_k\}$ is a strictly increasing sequence of positive integers.
Now,
$d(x, x_n) \leq d\left(x, x_{n_{n'}}\right) + d\left(x_{n_{n'}}, x_n\right) < d\left(x, x_{n_{n'}}\right) + \varepsilon)$.
whenever $n, n' \geq m(\varepsilon)$
Taking $n' \to \infty$ we get
$d(x, x_n) < \varepsilon$.
whenever $n, n' \geq m(\varepsilon)$.
Therefore, the sequence $\{x_n\}$ converges to x.

3.6 BANACH SPACE

A normed linear space $(X, \|\ \ \|)$ is said to be a Banach space if X is complete metric space under a metric d induced by the norm on X. Here,

$$d(x, y) = \|x - y\| \forall x, y \in X.$$

In other words, a complete normed linear space $(X, \|\ \ \|)$ is a Banach space.

A complete normed linear space is called a Banach space; i.e., we have a vector space on which we have defined a norm that gives you a metric topology called the norm topology, and if this topology is complete then the normed linear space is called a Banach space.

Examples:

(1) $(R, |\quad|)$ is a Banach space, where $|\quad|$ = absolute value.

(2) $(\mathbb{C}, |\quad|)$ is a Banach space, where $|\quad|$ = absolute value.

3.7 EXAMPLES

1. The linear space $\mathbb{R}$ and C of of real and complex numbers are Banach spaces under the norm $\|x\| = |x|, \forall x \in \mathbb{R}$ or C as the case may be.

Solution:

$\mathbb{R}$ is a normed linear space, since:

i. Since each $\|x\| \geq 0$ implies that $|x| \geq 0\ \forall x \in \mathbb{R}$.

ii. $\|x\| = 0 \Leftrightarrow |x| = 0 \Leftrightarrow x = 0, \forall x \in \mathbb{R}$

iii. $\|x + y\| = |x + y| \leq |x| + |y| = \|x\| + \|y\|, \forall x, y \in \mathbb{R}$

iv. $\|\alpha x\| = |\alpha x| = |\alpha|\|x\|, \alpha$ being real or complex.

Similarly C is a normed linear space, since:

i. Since each $\|x\| \geq 0$ implies that $|x| \geq 0\ \forall x \in C$.

ii. $\|x\| = 0 \Leftrightarrow |x| = 0 \Leftrightarrow x = 0, \forall x \in C$

iii. $\forall x, y \in C$ and $\bar{x}, \bar{y}$ being their conjugates (complex),

We have

$$|x + y|^2 = (x + y)\overline{(x + y)}$$

$$= (x + y)(\bar{x} + \bar{y}) = x\bar{x} + y\bar{y} + x\bar{y} + \bar{x}y$$

$$\leq |x|^2 + |y|^2 + 2|x\bar{y}|$$ [By properties of complex quantities]

$= |x|^2 + |y|^2 + 2|x||y|$ as $|\bar{y}| = |y|$

$= (|x| + |y|)^2.$

Giving $|x+y| \leq \|x\|+\|y\|$

iv. $\|\alpha x\| = |\alpha x| = |\alpha x| = |\alpha||\|x\|,\ \alpha$ being real or complex.

Since every convergent sequence in a normed linear space being a Cauchy sequence, the real $\mathbb{R}$ and C normed linear space is complete and hence a Banach space.

2. The linear space $\mathbb{R}^n$ and C^n of all $n-$tuples $(x_1, x_2, \dots., x_n)$ of real and complex numbers are Banach spaces under the norm

$$\|x\| = \left\{\sum_{i=1}^{n} |x_i|^2\right\}^{1/2}$$

[Usually called Euclidean and unitary spaces respectively].

Solution:

i. Since each $|x_i| \geq 0$ we have $\|x\| \geq 0$ and

$$\|x\| = 0 \ \Leftrightarrow \sum_{i=1}^{n} |x_i|^2 = 0 \ \Leftrightarrow x_i = 0, \text{for all } 1 \leq i \leq n$$

$$\Leftrightarrow x = (x_1, x_2, \dots \dots . x_n) = (0,0,0 \dots \dots ..0, \dots \dots 0) = 0$$

$$\Leftrightarrow x = 0$$

ii. $x = (x_1, x_2, \dots \dots . x_n), y = (y_1, y_2, \dots \dots . y_n) \in \mathbb{C}^n (or\ \mathbb{R}^n);$

$x + y = (x_1 + y_1, x_2 + y_2, \dots \dots . x_n + y_n), \in \mathbb{C}^n;$

Then,

$$\|x+y\|^2 = \sum_{i=1}^{n} |x_i + y_i|^2 = \sum_{i=1}^{n} |x_i + y_i|\,|x_i + y_i|$$

$$\leq \sum_{i=1}^{n} |x_i + y_i|\,(|x_i| + |y_i|) \text{[Since, } |x_i + y_i| \leq |x_i| + |y_i|]$$

$$= \sum_{i=1}^{n} |x_i + y_i|\,|x_i| + \sum_{i=1}^{n} |x_i + y_i|\,|y_i|$$

$$\leq \|x+y\|\|x\| + \|x+y\|\|y\| [\sum_{i=1}^{n} |x_i + y_i| \leq \|x\| + \|y\|]$$

$$= \|x+y\|(\|x\| + \|y\|)$$

or, $\|x+y\|^2 \leq \|x+y\|(\|x\|+\|y\|)$

or, $\|x+y\| \leq \|x\|+\|y\|$. For $\|x+y\| \neq 0$.

iii. For all $x\epsilon\mathbb{C}^n$, for all $\alpha\epsilon\mathbb{C}$

$$\|\alpha x\| = \sqrt{\sum_{i=1}^{n}|\alpha x_i|^2} = |\alpha|\{\sum_{i=1}^{n}|x_i|^2\}^{1/2} = |\alpha|\|x\|.$$

Therefore, $\|\alpha x\| = |\alpha|\|x\|$for all $x\epsilon\mathbb{C}^n, \forall\alpha \in \mathbb{C}$.

This proves that $\mathbb{R}^n$ or C^n are normed linear space.

Now, we show that the completeness of $\mathbb{R}^n$ or C^n.

Let $< x_1, x_2, \dots., x_n >$ be a Cauchy sequence in $\mathbb{R}^n$ or C^n.

Since x_m is an $n -$ tuple of complex (or real) numbers, we shall write,

$$x_m = \left(x_1^{(m)}, x_2^{(m)}, \dots \dots . x_n^{(m)}\right).$$

So that $x_k^{(m)}$ is the k^{th} coordinate of x_m.

Let $\in> 0$ be given, since $< x_m >$ is a Cauchy sequence, there exists a positive integer m_0, such that,

$$l, m \geq m_0 \Rightarrow \|x_m - x_l\| < \varepsilon$$

it implies that, $\|x_m - x_l\|^2 < \varepsilon^2$

$$\Rightarrow \sum_{i=1}^{n}\left|x_i{}^{(m)} - x_i{}^{(l)}\right| < \varepsilon^2$$

…………………………..[1]

$$\Rightarrow \left|x_i{}^{(m)} - x_i{}^{(l)}\right| < \varepsilon^2 \ (i = 1,2, \dots \dots, n)$$

$$\Rightarrow \left|x_i{}^{(m)} - x_i{}^{(l)}\right| < \varepsilon$$

Hence $\langle x_i^{<m>}\rangle_{m=1}^{\infty}$ is a Cauchy sequence of complex (or real) numbers for each fixed but arbitrary i.

Since C(or $\mathbb{R}$) is complete, each of these sequences converges to a point, say z_i in C(or $\mathbb{R}$) so that,

$$\lim_{m\to\infty} x_i{}^{(m)} = z_i \ (i = 1,2, \dots n)$$

……………………………[2]

Now, we show that the Cauchy sequence $< x_m >$ converges to the point $z = (z_1, z_2, \dots \dots .. z_n) \in C^n$ or $\mathbb{R}^n$.

To prove this let $l \to \infty$ in [1]. Then, by [2] we have,

$$\left|x_i^{(m)} - z_i\right|^2 < \varepsilon^2$$

$$\Rightarrow \|x_m - z\|^2 < \varepsilon^2$$

$$\Rightarrow \|x_m - z\| < \varepsilon$$

It follows that the Cauchy sequence $< x_m >$ converges to the point $z \in C^n$ or $\mathbb{R}^n$.

Hence, C^n or $\mathbb{R}^n$ are complete spaces and consequently they are Banach spaces.

Example 3:

Let p be a real number such that $1 \leq p < \infty$. Show that the space l_p^n of all n −tuples of scalars with the norm defined by:

$$\|x\|_p = \left\{\sum_{i=1}^{n} |x_i|^p\right\}^{\frac{1}{p}}$$

is a Banach space.

Solution:

Let $x = (x_1, x_2, \dots .. x_n)$ and $y = (y_1, y_2, \dots .. y_n)$ and let α be any scalar.

Then it is understood here that l_p^n is a linear space with respect to the operations,

$$x + y = (x_1 + y_1, \dots \dots .. x_n + y_n)$$

and

$\alpha x = (\alpha x_1, \alpha x_2, \dots .. \alpha x_n)$.

We now show that l_p^n is a normed linear space.

(i) $\|x\|_p \geq 0$, obvious since $|x_i| \geq 0$ for each i.

(ii) $\|x\|_p = 0 \Leftrightarrow \{\sum_{i=1}^{n} |x_i|^p\}^{\frac{1}{p}} = 0$

$$\Leftrightarrow \sum_{i=1}^{n} |x_i|^p = 0 \Leftrightarrow |x_i| = 0 (i = 1,2, \dots \dots n)$$

$$\Leftrightarrow x_i = 0, i = 1,2, \dots \dots, n$$

$$\Leftrightarrow x = (x_1, x_2, \dots \dots x_n) = 0.$$

(iii) $\|x + y\|_p \le \|x\|_p + \|y\|_p$, by Minkowski's inequality.

(iv) $\|\alpha x\|_p = \|\alpha(x_1, x_2, \dots \dots x_n)\|_p = \|(\alpha x_1, \alpha x_2, \dots \dots \alpha x_n)\|_p$

$$= \left\{\sum_{i=1}^{n} |\alpha x_i|^p\right\}^{\frac{1}{p}} = \left\{\sum_{i=1}^{n} |\alpha|^p \, |x_i|^p\right\}^{\frac{1}{p}} = \left\{|\alpha|^p \sum_{i=1}^{n} |x_i|^p\right\}^{\frac{1}{p}}$$

$$= |\alpha| \|x\|_p$$

Thus l_p^n is a normed linear space.

Again to show that l_p^n is complete.

Let $< x_m >_{m=1}^{\infty}$ be a Cauchy sequence in l_p^n.

Since, each x_m is an n −tuple of scalars,

$$x_m = (x_1^m, x_2^m \dots \dots x_n^m).$$

Let $\varepsilon > 0$ be given.

Since $< x_m >_{m=1}^{\infty}$ is a Cauchy sequence, there exists a positive integer m_0 such that,

$l, m \ge m_0 \quad \Longrightarrow \|x_m - x_l\|_p < \varepsilon$

$$\Longrightarrow \|x_m - x_l\|_p^p < \varepsilon^p \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots [3]$$

$$\Longrightarrow \sum_{i=1}^{n} \left|x_i^{(m)} - x_i^{(l)}\right|^p < \varepsilon^p \; (i = 1,2, \dots \dots n)$$

$$\Longrightarrow \left|x_i^{(m)} - x_i^{(l)}\right| < \varepsilon$$

This shows that for fixed but arbitrary i, the sequence $< x_i^{(m)} >_{m=1}^{\infty}$ is a Cauchy sequence in C or $\mathbb{R}$ so that,

$$\lim_{m\to\infty} x_i^{(m)} = z_i, (i = 1,2 \dots., n) \dots \dots \dots \dots . [4]$$

It will now be shown that the Cauchy sequence $< x_m >$ converges to the point $x = (z_1, z_2, \ldots\ldots z_n) \in l_p^n$.

To prove this, we let $l \to \infty$ in [3].

Then by [4], for $m \geq m_0$. We obtain,

$$\sum_{i=1}^{n} \left| x_i^{(m)} - z_i \right|^p < \varepsilon^p$$

$$\Rightarrow \|x_m - z\|_p^p < \varepsilon^p$$

$$\Rightarrow \|x_m - z\| < \varepsilon.$$

It follows that the Cauchy sequence $< x_m >$ converges to $z \in l_p^n$.

Hence l_p^n is complete therefore it is a Banach spaces.

Example 4: Consider the linear space of all $n-$ tuples $x = (x_1, x_2, \ldots\ldots x_n)$ of scalars and define the norm by

$$\|x\|_\infty = \max\{|x_1|, |x_2|, \ldots |x_n|\}.$$

This space is denoted by the symbol l_∞^n. Show that $(l_\infty^n, \|.\|_\infty)$ is a Banach space.

Solution : We first prove that l_∞^n is a normed linear space.

(i) $\|x\|_\infty \geq 0$, obvious since $|x_n| \geq 0$ for each n.

(ii) $\|x\|_\infty = 0 \Leftrightarrow \max\{|x_1|, |x_2|, \ldots |x_n|\} = 0$

$$\Leftrightarrow |x_1| = 0, |x_2| = 0, \ldots\ldots |x_n| = 0$$

$$\Leftrightarrow x_1 = 0, x_2 = 0, \ldots\ldots x_n = 0$$

$$\Leftrightarrow x = (x_1, x_2, \ldots\ldots x_n) = 0.$$

$$\Leftrightarrow x = 0.$$

(iii) Let $x = (x_1, x_2, \ldots\ldots x_n), y = (y_1, y_2, \ldots\ldots y_n)$

Then $\|x + y\|_\infty = \max\{|x_1 + y_1|, |x_2 + y_2|, \ldots |x_n + y_n|\}$

$\leq \max\{|x_1| + |y_1|, |x_2| + |y_2|, \ldots |x_n| + |y_n|\}$

$\leq \max\{|x_1|, |x_2|, \ldots\ldots |x_n|\} + \max\{|x_1|, |x_2|, \ldots\ldots |x_n|\}$

$\leq \max\{|x_1|, |x_2| \ \dots |x_n|\} + \max\{|y_1|, |y_2| \ \dots |y_n|\}$

$= \|x\|_\infty + \|y\|_\infty$

(iv) $\|\alpha x\|_\infty = \max\{|\alpha x_1|, |\alpha x_2|, \dots \dots |\alpha x_n|\}$

$= \max\{|\alpha||x_1|, |\alpha||x_2|, \dots \dots |\alpha||x_n|\}$

$= |\alpha| \max\{|x_1|, |x_2|, \dots \dots |x_n|\}$

$= |\alpha| \|x\|_\infty.$

Hence l_∞^n is a normed linear space.

We now show that it is a complete space.

Let $< x_m >_{m=1}^{\infty}$ be a Cauchy sequence in l_∞^n.

Since, each x_m is an n −tuple of scalars,

$$x_m = (x_1^m, x_2^m \dots \dots x_n^m).$$

Let $\varepsilon > 0$ be given.

Then there exists a positive integer m_0 such that,

$l, m \geq m_0 \quad \Rightarrow \|x_m - x_l\|_p < \varepsilon$

$\Rightarrow \max\left\{\left|x_1^{(m)} - x_1^{(l)}\right|, \left|x_2^{(m)} - x_2^{(l)}\right|, \dots \left|x_n^{(m)} - x_n^{(l)}\right|\right\} < \varepsilon$…………..[5]

$$\Rightarrow \left|x_i^{(m)} - x_i^{(l)}\right| < \varepsilon, i = 1,2, \dots \dots \dots, n.$$

This shows that for fixed but i, the sequence $< x_i^{(m)} >_{m=1}^{\infty}$ is a Cauchy sequence of complex or real numbers.

Since C or $\mathbb{R}$ is complete, it must converges to some $z_i \in C$ or $\mathbb{R}$.

We assert that the Cauchy sequence $< x_n >$ converges to the point $z = (z_1, z_2, \dots \dots z_n)$.

To prove this, we let $l \to \infty$ in [5].

Then, for $m \geq m_0$.

We obtain $\|x_m - z\| < \varepsilon$.

Thus it follows that the Cauchy sequence $< x_m >$ converges to $z \in l_\infty^n$.

Hence l_∞^n is complete.

Therefore l_∞^n is a Banach space.

Example 5: If $C(X)$ be a linear space of all bounded continuous scalar valued function defined on a topological space X. Then show that $C(X)$ is a Banach space under the norm

$$\|f\| = \sup\{|f(x)|: x \in X\}, f \in C(X).$$

Solution: Given that $C(X)$ is a linear space, means $C(X)$ is linear under the operations of vector addition and scalar multiplication i.e.,
$f, g \in C(X)$ and α being a scalar, we knows that,

$$(f+g)(x) = f(x) + g(x),$$
$$(\alpha f)(x) = \alpha f(x).$$

We now show that $C(X)$ is normed linear space.

Solution.

(i) Since $|f(x)| \geq 0 \forall x \in X$,

$$\|f(x)\| \geq 0$$

(ii) $\|f\| = 0 \Leftrightarrow \sup\{|f(x)|: x \in X\} = 0$

$$\Leftrightarrow |f(x)| = 0 \forall x \in X$$
$$\Leftrightarrow f(x) = 0$$
$$\Leftrightarrow f \text{ is a zero function.}$$

(iii) Then$\|f\| = \sup\{|(f+g)(x)|: x \in X\}$

$$= \sup\{|f(x) + g(x)|: x \in X\}$$
$$\leq \sup\{|f(x)| + |g(x)|: x \in X\}$$
$$\leq \sup\{|f(x)|: x \in X\} + \sup\{|g(x)|: x \in X\}$$
$$= \|f\| + \|g\|$$

(iv) $\|\alpha f\| = \sup\{|(\alpha f)(x)| : x \in X\}$

$$= \sup\{|\alpha f(x)| : x \in X\} = \sup\{|\alpha||f(x)| : x \in X\}$$
$$= |\alpha| \sup\{|f(x)| : x \in X\} = |\alpha| \, \|f\|.$$

Hence $C(X)$ is normed linear space.

Now for proving $C(X)$ is complete..

Let $< f_n >_{n=1}^{\infty}$ be a Cauchy sequence in $C(X)$.

Then for given $\varepsilon > 0$,

Then there exists a positive integer n such that,

$m, n \geq m_0 \quad \Rightarrow \|f_m - f_n\|_p < \varepsilon$

$\Rightarrow \sup\{|f_m(x) - f_n(x)| : x \in X\} < \varepsilon$…………..[6]

$\Rightarrow \{|f_m(x) - f_n(x)|\} < \varepsilon \ \forall x \in X.$

But this is the Cauchy condition for uniform convergence of the sequence of bounded continuous scalar valued function.

Hence the sequence $< f_n >$ must converge to bounded continuous function f on X.

It implies that $C(X)$ is complete and hence it is a Banach space.

3.8 SUMMARY

Present unit is presentation of the topic Continuous at a point, Cauchy Sequence, Completeness and Banach Space. The main focus is in this unit on Banach Space. The above concepts discussed with the help of Examples and Main Results.

3.9 GLOSSARY

i. **Metric space:** Let $X \neq \emptyset$ be a set then the metric on the set X is defined as a function $d: X \times X \to [0, \infty)$ such that some conditions are satisfied.

ii. **Vector space: -** Let V be a nonempty set with two operations

(i) **Vector addition:** If any $u, v \in V$ then $u + v \in V$

(ii) **Scalar Multiplication:** If any $u \in V$ and $k \in F$ then $ku \in V$

Then V is called a vector space (over the field F) if the following axioms hold for any vectors if the some conditions hold.

iii. **Normed space:-** Let X be a vector space over scalar field K. A *norm* on a (real or complex) vector space X is a real-valued function on X ($\|x\|: X \to K$) whose value at an $x \in X$ is denoted by $\|x\|$ and which has the four properties here x and y are arbitrary vectors in X and α is any scalar.

CHECK YOUR PROGRESS

1. Let V be a Banach space. Define

$$X = C([0,1]: V) = \{f: [0,1] \longrightarrow V | f \text{ } is \text{ } continuous\}.$$

Define, for $f \in X$,

$$\|f\| = \sup_{t \in [0,1]} \|f(t)\|_V$$

Which of the following statements are true?

a) This defines a norm on X.

b) We have $\|f\| = \sup_{t \in [0,1]} \|f(t)\|_V$

c) X is a Banach space with this norm.

2.

Let $C[a,b]$ be the space of all complex valued continuous functions on $[a,b]$. Under which of the following norms, $C[a,b]$ is a Banach space?

(a) $\|f\| = (\int_a^b |f(t)|^2 dt)^{1/2}$

(b) $\|f\| = \int_a^b |f(t)| dt$

(c) $\|f\| = (\int_a^b |f(t)|^3 dt)^{1/3}$

(d) None of these.

3.

A complete normed space is known as a :

(a) Hilbert space

(b) Compact space

(c) Banach space

(d) Euclidean space

4.

Which of the following is a Banach space?

(a) Space of all polynomial functions on $[a, b]$ with the supremum norm

(b) Space of all continuous functions on $[a, b]$ with the supremum norm

(c) Space of all polynomial functions on $[a, b]$ with the p-norm

(d) Space of all continuous functions on $[a, b]$ with the p-norm

5.

Which of the following subspaces of ℓ_∞ is not a Banach space?

(a) c

(b) c_0

(c) s^*

(d) ℓ_p

6.

Which of the following is not a Banach space?

(a) Linear space of all n-tuples $x = (a_1, a_2, ..., a_n)$ with $\|x\| = \max_i |a_i|$.

(b) Linear space of all 2-summable sequences $x = (a_1, a_2, ...)$ with $\|x\| = (\sum_{i=1}^{\infty} |a_i|^2)^{1/2}$.

(c) Linear space of all bounded sequences $x = (a_1, a_2, ...)$ with $\|x\| = \sup_i |a_i|$.

(d) Linear space of all continuous functions on $[0, 1]$ with $\|f\| = \int_0^1 |f(t)|dt$.

7.

Consider the statements:

(i) Every normed space is complete.

(ii) Every normed space can be identified as a dense subspace of a complete normed space.

(a) Only (i) is true

(b) Only (ii) is true

(c) Both (i) and (ii) are true

(d) Neither (i) nor (ii) are true.

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i. E. Kreyszig, (1989), *Introductory Functional Analysis with applications*, John Wiley and Sons.

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3.11 TERMINAL QUESTIONS

1. What is meant by Banach space?

………………………………………………………………..

2. What is the difference between complete space and Banach space?

…………………………………………………………

3. Show that the space $\mathcal{C}$(vector space of all convergent sequence of complex number) is a Banach space?

…………………………………………………………

4. Show that the $(\mathcal{C}_0, \|\ \ \|_\infty)$ (the space of all sequences converging to zero, with sup norm) is a Banach space?

……………………………………………………………

5. Show that the $\left(\mathcal{C}_{00}, \|\ \ \|_p\right)$ where $\mathcal{C}_{00} = \{(x_n): x_n = 0, all\ but\ finitely\ many\ n\}$ is not a Banach space?

3.12 ANSWERS

CHECK YOUR PROGRESS

1. c
2. d
3. c
4. b
5. c
6. d
7. b

UNIT 4:

FINITE DIMENSIONAL SPACES

CONTENTS:

4.1 INTRODUCTION

In previous units we have studied about Metric Space, Vector Space, Normed Space and Banach Space. Now we are familiar with functional analysis. For continuation of the study of functional analysis we need more study. In this unit we are explaining about quotient spaces, subspace of a normed space, subspace of a Banach space and finite dimensional normed space.

Since we know that a linear space X is said to be finite dimensional space if there is a finite basis for X. A linear space which is not a finite dimensional space is called an infinite dimensional space.

We begin with some questions.

- What is the dimension of vector space $\mathbb{C}$ over $\mathbb{R}$?

  Solution. $\mathbb{C}$ is a vector space over $\mathbb{R}$. Since every complex number is uniquely expressible in the form a + bi with a, b ∈ R we see that (1, i) is a basis for $\mathbb{C}$ over $\mathbb{R}$. Thus the dimension is two.

- What is the dimension of vector space $F[x]$ of all polynomials over a field F?

  Solution. Infinite

4.2 OBJECTIVES

After studying this unit, learner will be able to

i. Described the concept of quotient space.

ii. Defined the concept of subspace of a normed and Banach space.

iii. Explained the topic of finite dimensional normed space.

4.3 QUOTIENT (FACTOR) SPACES

If M be a subspace of a vector space N, then their exists an equivalence relation between any two vectors

$x, y \in N$ $i.e.$, $x \sim y$ iff $x - y \in M$, since this relation is:

Reflexive i.e., $x \sim y$ as $x - y = 0 \in M$.

Symmetric i.e., $x \sim y \Rightarrow y \sim x$ as $x - y \in M$.

$\Rightarrow$ as $(x - y) = y - x \in M$.

Transitive i.e., $x \sim y, y \sim z \Rightarrow x \sim z$ as

$x - y \in M$ and $y - z \in M \Rightarrow x - y + y - z = x - z \in M$.

Therefore vectors x, y being equivalent under $'\sim' \Rightarrow x - y \in M$.

Thus N is divided into mutually disjoint equivalence classes.

We denote the set of all such equivalence classes by $\frac{N}{M}$.

Let $[x]$ denote the equivalence class which contains the element x. Thus,

$$[x] = \{y : y \sim x\} = \{y : y - x \in M\}$$
$$= \{y : y - x = m \; for \; some \; m \in M$$
$$= \{y : y = x + m \; for \; some \; m \in M\} = \{x + m : m \in M\}.$$

Thus $[x]$ is the set of all sums of x and element of M.

The set $[x]$ is called the coset of M determined by x and is usually written as $x + M$.

In $\frac{N}{M}$, we define addition and scalar multiplication by,

$$(x + M) + (y + M) = (x + y) + M; x, y \in N$$

$\alpha(x + M) = (\alpha x) + M, \alpha \in F$ over which N is defined.

Here $\frac{N}{M}$ is a vector (linear) space with respect to addition and scalar multiplication.

Also N is a normed linear space and exhibits' a norm for $\frac{N}{M}$. The zero element of $\frac{N}{M}$ is $0 + M = M$.

The set of all such equivalence classes $\{x + m : m \in M\}$ referred as $\frac{N}{M}$ is known as the Factor space or Quotient space of N with respect to N.

4.3.1 THEOREM

Theorem: If M be a closed subspace of a normed linear space N and if the norm of a coset $x + M$ is the quotient space $\frac{N}{M}$ is defined by

$$\|x + M\| = \inf\{\|x + m\| : m \in M\} \ldots\ldots..(1)$$

Then $\frac{N}{M}$ is a normed linear space. Also if N is complex (Banach space), then so is $\frac{N}{M}$.

Proof.

Now for $\frac{N}{M}$ is a normed linear space,

i. Since $\|x + m\|$ is non-negative real number and every set of non-negative real numbers is bounded below, it follows that $\inf\{\|x + m\| : m \in M\}$ exists and is non-negative, that is

$$\|x + m\| \geq 0 \forall x \in N.$$

ii. Let $x + M = M \left(the\ zero\ element\ of\ \frac{N}{M}\right)$. Then $x \in M$.

Hence $\|x + M\| = \inf\{\|x + m\| : m \in M, x \in M\}$

$$= inf\{\|y\| : y \in M\} = 0$$

$[\because M\ being\ a\ subspace\ contains\ zero\ vector\ whose\ norm\ is\ real\ number\ 0]$

Thus $x + m = M \Rightarrow \|x + M\| = 0$.

Conversely, we have

$$\|x + M\| = \inf\{\|x + m\|: m \in M\} = 0$$

$\Longrightarrow$there exists a sequence $< m_k >_{k=1}^{\infty}$ in M.

Such that $\|x + m_k\| \to 0$ as $k \longrightarrow \infty$.

$\Longrightarrow \lim_{n\to\infty} m_k = -x$.

$\Longrightarrow -x \in M$

[Since M is closed and $< m_k >$ is sequence in M converging to $-x$]

$\Longrightarrow x \in M$ [Since M is a subspace].

$\Longrightarrow x + M = M$ [the zero element of $\frac{N}{M}$].

Thus we have shown that

$$\|x + M\| = 0 \Rightarrow x + M = M(the\ zero\ element\ of\ \ N/M).$$

iii. Let $x + M,\ y + M \in \frac{N}{M}$, then

$\|(x + M) + (y + M)\| = \|(x + y) + M\|$

(by definition of addition of coset)

$= \inf\{\|x + y + m\|: m \in M\} \ldots\ldots..(1)$

$= \inf\{\|x + y + m\|: m \in M, m' \in M\ \} \ldots\ldots..(2)$

(Since M is a subspace, the sets in (1) and (2) are the same).

$= \inf\{\|(x + m) + (y + m')\|: m, m' \in M\ \}$

$\leq$inf $\{\|x + m\| + \|y + m'\|: m, m' \in M\}$

[Using iii for N, since $x + m, y + m' \in N$]

$= \inf\{\|x + m\|: m \in M\ \} +\ \inf\{\|x + m'\|: m' \in M\ \}$

$= \|x + M\|$+$\|y + M\|$.

iv. $\|\alpha(x + M)\| = \inf\{\|\alpha x + m\|: m \in M\}$

(since $\alpha(x + M) = \alpha x + M$ in $\frac{N}{M}$)

$= \inf\{\|\alpha x + m\|: m \in M\}$ if $\alpha \neq 0$.

$= |\alpha|\ \inf\{\|x + m\|: m \in M\}$

$= |\alpha| \|x + M\|.$

For $\alpha = 0$, the results is obvious.

Hence, $\frac{N}{M}$ is a normed linear space.

We now prove that if N is complete, then so is $\frac{N}{M}$.

Suppose that $< x_n + M >$ is a Cauchy sequence in $\frac{N}{M}$.

Then to show that $< x_n + M >$ is convergent, it is sufficient to prove that this sequence has convergent subsequence.

We can easily find a subsequence of the original Cauchy sequence for a fixed n such that,

$$\|(x_1 + M) - (x_2 + M)\| < \frac{1}{2}$$

$$\|(x_2 + M) - (x_3 + M)\| < \frac{1}{2^2}$$

……………………………….

………………………………………

…………………………………………..

$$\|(x_n + M) - (x_{n-1} + M)\| < \frac{1}{2^n}$$

We prove that this sequence is convergent in $\frac{N}{M}$.

We begin by choosing any vector y_1 in $x_1 + M$, and we select y_2 in $x_2 + M$ such that $\|y_1 - y_2\| < \frac{1}{2}$.

We next select a vector y_3 in $x_3 + M$. Such that $\|y_n - y_{n+1}\| < \frac{1}{2^n}$.

Thus for $m < n$, we have

$$\|y_m - y_n\| = \|(y_m - y_{m+1}) + (y_{m+1} - y_{m+2}) + \cdots (y_{n-1} - y_n)\|$$

$$\leq \|y_m - y_{m+1}\| + \|y_{m+1} - y_{m+2}\| + .. + \|y_{n-1} - y_n\|$$

$$< \frac{1}{2^m} + \frac{1}{2^{m+1}} + \cdots \frac{1}{2^{n-1}}$$

$$= \frac{1}{2^m}\left[1 + \frac{1}{2} + .. + \frac{1}{2^{n-m-1}}\right]$$

$$= \frac{1}{2^m}\left[\frac{1 - \left(\frac{1}{2}\right)^{n-m}}{1 - \frac{1}{2}}\right] = \frac{1}{2^{m-1}}\left[1 - \frac{1}{2^{n-m}}\right] < \frac{1}{2^{m-1}} \to 0 \; as \; m \longrightarrow \infty,$$

which follows that $\langle y_n \rangle$ is a Cauchy sequence in N.

Since N is complete, there exists a vector y in N such that $y_n \longrightarrow y$.

It now follows from

$\|(y_n + M) - y + M)\| \leq \|y_n - y\| \dashrightarrow 0$ as $n \longrightarrow \infty$,

that $y_n + M \longrightarrow y + M$ it means $y_n + M$ converges to $y + M$ in $\frac{N}{M}$.

Hence $\frac{N}{M}$ is complete.

4.4 SUBSPACE

4.4.1 SUBSPACE OF A NORMED SPACE

A subspace Y of a normed space X is a subspace of X considered as a vector space, with the norm obtained by restricting the norm on X to the subset Y. This norm on Y is said to be induced by the norm on X. If Y is closed in X, then Y is called a closed subspace of X.

4.4.2 SUBSPACE OF A BANACH SPACE

A subspace Y of a Banach space X is a subspace of X considered as a normed space.

4.5. FINITE DIMENSIONAL NORMED SPACES

4.5.1 LEMMA

Let $\{x_1, \dots x_n\}$ be a linearly independent set of vectors in a normed space X $(of\ any\ dimension)$, then there is a number $c > 0$ such that for every choice of scalar $\alpha_1, \dots \dots \dots \alpha_n$ we have,

$$\|\alpha_1 x_1 + \cdots \dots \dots \dots \dots + \alpha_n x_n\| \geqq c(|\alpha_1| + \cdots |\alpha_n|) \dots\dots\dots\dots (2)$$

Proof. We write $s = |\alpha_1| + \cdots |\alpha_n|$.

If $s = 0$,

all α_j are zero, so that (2) holds for any c.

Let $s > 0$.

Then (2) is equivalent to the inequality which we obtain from (2) by dividing by s and writing $\beta_j = \frac{\alpha_j}{s}$, that is,

$$\|\beta_1 x_1 + \cdots \dots \dots \dots \dots + \beta_n x_n\| \geqq c \quad (\sum_{j=1}^{n} |\beta_j| = 1)$$

Hence it satisfy to prove the existence of a $c > 0$ such that (2) holds for every n −tuple of scalars $\beta_1 \dots \dots \dots \dots \dots, \beta_n$ with $\sum_{j=1}^{n} |\beta_j| = 1$.

Suppose that this is false.

Then there exists a sequence (y_m) of vectors,

$$y_m = \beta_1^{(m)} x_1 + \cdots \dots \dots \dots \dots \beta_n^{(m)} x_n \quad (\sum_{j=1}^{n} |\beta_j^{(m)}| = 1).$$

Such that $\|y_m\| \to 0$ as $m \longrightarrow \infty$.

Now we reason as follows.

Since $\sum_{j=1}^{n} |\beta_j^{(m)}| = 1$, we have $|\beta_j^{(m)}| \leqq 1$.

Hence for each fixed j the sequence

$\left(\beta_j^{(m)}\right) = \left(\beta_1^{(m)}, \beta_2^{(m)}, \dots \dots ..\right)$ is bounded.

Consequently, by the **Bolzano – Weierstrass theorem**, $\left(\beta_j^{(m)}\right)$ has a convergent subsequence.

Let β_1 denote the limit of that subsequence, and let , $\left(y_{1,m}\right)$ has a subsequence of $\left(y_{2,m}\right)$ for which the corresponding subsequence of scalars $\beta_2^{(m)}$ converges.

Let β_2 denote the limit.

Continuing in this way, after n steps we obtain a subsequence $\left(y_{n,m}\right) = \left(y_{n,1}, y_{n,2} \dots.\right)$ of (y_m) whose terms are of the form.

$y_{n,m} = \sum_{i=1}^{n} \gamma_j^{(m)} x_j \left(\sum_{j=1}^{n} \left|\gamma_j^{(m)}\right| = 1\right),$

with scalars $\gamma_j^{(m)}$ satisfying $\gamma_j^{(m)} \longrightarrow \beta_j$ as $m \to \infty$.

Hence, as $m \to \infty. y_{n,m} \longrightarrow y = \sum_{j=1}^{n} \beta_j x_j$.

Where $\sum\left|\beta_j\right| = 1$, so that not all β_j can be zero.

Since $\{x_1, \dots x_n\}$ be a linearly independent set of vectors we thus have $y \neq 0$.

On the other hand, $y_{n,m} \longrightarrow y$ implies $\left\|y_{n,m}\right\| \to \|y\|$, by the continuity of the norm.

Since $\|y_m\| \to 0$.

Hence $\|y\| = 0$. So that $y = 0$. (Second property of norm)

This contradicts $y \neq 0$, and the lemma is proved.

4.5.2 THEOREMS

Theorem 1: Every finite dimensional subspace Y of a normed space X is complete. In particular, every finite dimensional normed space is complete.

Proof. Consider an arbitrary Cauchy sequence (y_m) in Y and show that it is convergent in Y;

the limit will be denoted by y.

Let dim $Y = n$ and $\{e_1, \dots e_n\}$ any basis for Y.

Then each y_m has a unique representation of the form

$$y_m = \alpha_1^{(m)} e_1 + \cdots \dots \dots \dots \dots \alpha_n^{(m)} e_n.$$

Since (y_m) is a Cauchy sequence, for every $\varepsilon > 0$ there is N such that,

$$\|y_m - y_r\| < \varepsilon \text{ when } m, r > N.$$

Therefore from the above condition and using the **lemma 4.5.1** we have some $c > 0$,

$$\varepsilon > \|y_m - y_r\| = \left\| \sum_{j=1}^{n} \left(\alpha_j^{(m)} - \alpha_j^{(r)}\right) \right\| \geqq c \sum_{j=1}^{n} \left|\alpha_j^{(m)} - \alpha_j^{(r)}\right|,$$

when $m, r > N$. Division by when $c > 0$ gives

$$\left|\alpha_j^{(m)} - \alpha_j^{(r)}\right| \leq \sum_{j=1}^{n} \left|\alpha_j^{(m)} - \alpha_j^{(r)}\right| < \frac{\varepsilon}{c}, (m, r > N).$$

This shows that each of the n sequences,

$$\left(\alpha_j^{(m)}\right) = \left(\alpha_j^{(1)}, \alpha_j^{(2)}, \dots \dots\right) \qquad j = 1, \dots \dots \dots, n$$

is Cauchy in $\mathbb{R}$ or $\mathbb{C}$.

Hence it converges;

let α_j denote the limit.

Using these n limits $\alpha_1, \dots \dots . . \alpha_n$, we define,

$$y = \alpha_1 e_1 + \cdots + \alpha_n e_n.$$

It is clear that $y \in Y$.

Now,

$$\|y_m - y\| = \left\|\sum_{j=1}^{n}\left(\alpha_j^{(m)} - \alpha_j^{(r)}\right)\right\| \leqq \sum_{j=1}^{n}\left|\alpha_j^{(m)} - \alpha_j\right| \|e_j\|.$$

Since $\alpha_j^{(m)} \to \alpha_j$.

$\|y_m - y\| \to 0.$

It implies $y_m \to y$.

This shows that (y_m) was an arbitrary Cauchy sequence in Y.

This proves that Y is complete.

Theorem 2: Every finite dimensional subspace Y of a normed space X is closed in X.

Question:

Infinite dimensional subspace Y of a normed space X is closed in X?

Answer: Need not to be closed.

Example:

Let $X = C[0,1]$ and $Y =$span $\{x_0, x_1, \ldots \ldots\}$ where $x_i(t) = t^i$, so that Y is the set of all polynomials. Y is not closed in X.

- In finite dimensional vector space X is that all norms on X lead to the same topology for X.

4.6. EQUIVALENT NORMS

A norm $\|.\|$ on a vector space X is said to be equivalent to a norm $\|.\|_0$ on X if there are positive numbers a and b such that for all $x \in X$

We have

$$a\|x\|_0 \leqq \|x\| \leqq b\|x\|_0 \ldots\ldots\ldots\ldots\ldots\ldots\text{[E.N]}$$

- Equivalent norms on X define the same topology for X.
- Any two norms on a finite – dimensional space are equivalent but this does not extend to infinite-dimensional spaces.

4.6.1 THEOREM

On a finite – dimensional vector space X, any norm $\|.\|$ is equivalent to any other norm $\|.\|_0$.

Proof.

Let $X = n$.

Consider $\{e_1, e_2, \ldots. e_n\}$ any basis for X.

Then every $x \in X$ has a unique representation.

$$x = \alpha_1 e_1 + \cdots \quad + \alpha_n e_n.$$

Since from above Lemma,

[Let $\{x_1, \ldots x_n\}$ be a linearly independent set of vectors in a normed space X $(of\ any\ dimension)$, then there is a number $c > 0$ such that for every choice of scalar $\alpha_1, \ldots \ldots \ldots \alpha_n$ we have,

$\|\alpha_1 x_1 + \cdots \ldots \ldots \ldots \ldots + \alpha_n x_n\| \geqq c(|\alpha_1| + \cdots |\alpha_n|)$]

there is a positive constant c such that,

$\|x\| \geqq c(|\alpha_1| + \cdots |\alpha_n|)$.

The triangle inequality gives,

$$\|x\|_0 \leqq \sum_{j=1}^{n} |\alpha_j| \, \|e_j\|_0 \leq k \sum_{j=1}^{n} |\alpha_j|,$$

where $k = \ max_j \|e_j\|_0$.

Together, $a\|x\|_0 \leq \|x\|$,

Where $a = c/k > 0$.

The other inequality in [E.N] is now obtained by an interchange of the roles of $\|.\|$ and $\|.\|_0$ in the preceding element.

This theorem is of considerable practical importance. For instance, it implies that convergence or divergence of a sequence in a finite dimensional vector space does not depend on the particular choice of a norm on that space.

CHECK YOUR PROGRESS

Write True and False :

1. Dimension of $\mathbb{C}^n$ as a linear space over $\mathbb{R}$ is $2n$ True/False.
2. If E is finite dimensional linear space of dimension n, and F is a subset of E with m elements, where m < n, then F can be a basis of E. True/False.
3. Every finite dimensional normed space has a unique norm. True/False.
4. Every finite dimensional normed linear space is a Banach Space. True/False.
5. If E is finite dimensional linear space of dimension n, and F is a subset of E with m elements, where $m < n$, then F cannot be a basis of E. True/False.
6. Every complete subspace of a normed space is closed. True/False.
7. Let M be a closed subspace of a normed space N. Then the quotient space N/M is a Banach space if and only if: N is a Banach space. True/False.
8. For x, y in a normed space X, $|\|x+y\| - \|x-y\|| \geq 2\|y\|$. True/False.

9. All norms are equivalent on finite dimensional vector spaces.
True/False

10. The norms $\|f\|_1$ and $\|f\|_2$ are not equivalent in $C[0,1]$.
True/False

4.8 SUMMARY

Present unit is presentation of the concepts quotient spaces, subspace of a normed space, subspace of a Banach space and finite dimensional normed space. These concepts have been explained with the help of definitions, examples and theorems. The learners can understand the concepts in easy manner.

4.9 GLOSSARY

i. **Metric space:** Let $X \neq \emptyset$ be a set then the metric on the set X is defined as a function $d: X \times X \to [0, \infty)$ such that some conditions are satisfied.

ii. **Vector space: -** Let V be a nonempty set with two operations

(i) **Vector addition:** If any $u, v \in V$ then $u + v \in V$

(ii) **Scalar Multiplication:** If any $u \in V$ and $k \in F$ then $ku \in V$

Then V is called a vector space (over the field F) if the following axioms hold for any vectors if the some conditions hold.

iii. **Normed space:-** Let X be a vector space over scalar field K. A *norm* on a (real or complex) vector space X is a real-valued function on X ($\|x\|: X \to K$) whose value at an $x \in X$ is denoted by $\|x\|$ and which has the four properties here x and y are arbitrary vectors in X and α is any scalar.

iv. **Banach space:-** A complete normed linear space is called a Banach space.

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4.12 TERMINAL QUESTIONS

1. What is quotient space with example?
2. What is a subspace of a Banach space?
3. Is a normed linear space complete? ....
   ....
4. What are the characteristics of a Banach space?
   ....

4.13 ANSWERS

CHECK YOUR PROGRESS

1. True.
2. False.
3. False.
4. True.
5. True.
6. True.
7. True.
8. False.
9. True
10. True.

UNIT 5:

COMPACTNESS AND FINITE DIMENSION

CONTENTS:

5.1 INTRODUCTION

In previous units we have studied the concepts: quotient spaces, subspace of a normed space, subspace of a Banach space and finite dimensional normed space. These concepts have been explained with the help of definitions, examples and theorems. Compactness is one of the most fundamental mathematical notions. Because of that, after more than a century from its formal introduction, it still attracts great interest of researchers. Compactness is so widespread that it seems nigh to impossible to even briefly mention all the theories where it plays a crucial role.

5.2 OBJECTIVES

After studying this unit, learner will be able to

- **i.** Described the concept of Compactness
- **ii.** Explained the topic of F. Riesz's Lemma.

5.3 COMPACTNESS

First we recall the definition of cover and subcover.

Open cover of set: Let (X, d) be a metric space and G be a collection of open sets in X.If for each $x \in X$ there is a member $G_i \subseteq G$ such that $x \in G_i$, then G is called an opencover of X.

Subcover of set: A subcollection of G which is itself an open cover of$X$ is called a subcover (or subcovering).

Now we define compact set as

Compact Set: A metric space (X, d) is said to be compact if every open covering G of X has a finite subcovering, i.e., there is a finite subcollection $\{G_1, G_2, \dots, G_n\} \in G$ such that $X = \cup_{i=1}^{\infty} G_i$.

NOTE:

- A nonempty subset Y of X is said to be compact if it is a compact metric space with the metric induced on it by d.

- A nonempty subset Y is compact if every covering G of Y by relatively open sets of Y has a finite subcovering.

A metric space X is said to be compact if every sequence in X has a convergent subsequence.

A subset M of X is said to be compact if M is compact considered as a subspace of X, that is, if every sequence in M has a convergent subsequence whose limit is an element of M.

Example:

- The interval $(0,1)$ in the metric space $(\mathbb{R}, d)$, where d denotes the usual metric, is not compact. Now we will try to find an open covering such that given cover has no subcover.Consider the open covering $\left\{\left(\frac{1}{n}, 1\right) : n = 2, 3, \ldots\right\}$ of (0,1) . We observed there is no subcover for open cover. Mathematically $\cup_{n=2}^{\infty} S\left(0, 1-\frac{1}{n}\right) \supseteq S(0,1)$. But no finite subcollection of $\left\{S\left(0, 1-\frac{1}{n}\right) : n = 2, 3, \ldots\ldots\right\}$ covers open ball $S(0,1)$.
- Let Y be a finite subset of a metric space (X, d).Then$Y$ is compact.

Local Compactness:

A metric space X is said to be locally compact at every point of X has a compact neighbourhood.

- $\mathbb{R}^n$ and $\mathbb{C}^n$ are locally compact.

Relatively Compact:

A subset A of a metric space X is relatively compact if and only if every sequence of points in A has a cluster point in X. A space is compact if it is relatively compact in itself. An alternative definition is that A is relatively compact in X if and only if every open cover of X contains a finite subcover of A.

Theorem 1. Closed subsets of compact sets are compact.

Proof. Let Y be compact subset of metric space X.

Let $A \subseteq Y$ closed relative to Y and closed relative to X.

Now we will try to prove that A is compact

Let $G = \{G_\lambda : \lambda \in \Lambda\}$ be an oper cover of A.

Then the collection

$M = \{G_\lambda : \lambda \in \Lambda\} \cup \{X - A\}$forms an open cover of Y.

$Y$is compact⇒there is a finite sub-collection M^* of M which covers Y.

Therefore it also covers A.

If $X - A$ is a member of M^*, so we can remove it from M^* and it still remain open cover of A.

Thus Finite subcollection of G covers A.

Therefore A is compact.

Finite intersection property (F.I.P): .A collection F of sets in X is said to have the finite intersection property if every finite subcollection of F has a nonempty intersection.

Theorem2. Let (Y, d^*) be a subspace of metric space (X, d). Prove that Y is compact w.r.t metric d^* iff Y is compact w.r.t metric$d$ on X.

Proof. Let F_λ is $d^* -$ open cover of Y.

$\Rightarrow Y \subseteq \cup_\lambda F_\lambda$.

Again F_λ is $d^* -$ open cover

⇒there exists d-open G_λ such that $F_\lambda = G_\lambda \cap Y \subseteq G_\lambda$

⇒there exists d-open G_λ such that $\cup_\lambda F_\lambda \subseteq \cup_\lambda G_\lambda$

But $Y \subseteq \cup_i F_i$ and $Y \subseteq \cup_i G_i$

$\Rightarrow \{G_i\}$ is d-open cover of Y.

It is compact and therefore the cover G_i must have finite reducible subcover.

Let $\{G_{\lambda_k}: k = 1,2,3, \dots\}$ be subcover of G_λ.

$\Rightarrow Y \subseteq \cup_{\lambda=1}^{n} G_{\lambda_k}$

where

$$Y \cap Y \subseteq Y \cap (\cap_{k=1}^{n} G_\lambda) = \cap_{k=1}^{n} \left(A \cap G_{\lambda_r}\right) = \cup F_{\lambda_k}$$

$\Rightarrow Y \subseteq F_{\lambda_k}$ is ad^* $-$ open cover of A

$\Rightarrow$F is d^* $-$compact.

Converse

Let (Y, d^*) is a subspace of (X,d) and Y is d^* $-$compact

Now we prove that Y is d-compact.

Let G_λ is $d -$ open cover of $Y \Rightarrow Y \subseteq \cup_\lambda G_\lambda$.

Therefore $Y \cap Y \subseteq Y \cap (\cup_\lambda G_\lambda)$

It implies that $Y \subseteq \cup (Y \cap G_\lambda)$

Let $F_\lambda = G_\lambda \cap Y$ then $Y \subseteq \cup G_\lambda$

$\Rightarrow G_\lambda$ is d $-$open $\Rightarrow G_\lambda = G_\lambda \cap Y$ is d^* $-$open.

Therefore F_λ is a d^* $-$open cover of Y but $F_\lambda$is $d^* -$ compact.

Hence given cover is reducible to finite subcover. i.e.$\{F_{\lambda_k}: 1 \leq k \leq n\}$

$\Rightarrow Y \subseteq \cup_{k=1}^{n} F_{\lambda_k} = \cup_{k=1}^{n} (F_{\lambda_k} \cap Y)$

$\Rightarrow Y \subseteq \cup_{k=1}^{n} \left(G_{\lambda_k} \cap Y\right) = \cup_{k=1}^{n} G_{\lambda_k}$

$\Rightarrow Y \subseteq \cup_{k=1}^{n} G_\lambda$

$\Rightarrow G_\lambda$ is finite subcover of the cover G_λ.

Therefore Y is d $-$compact.

BolzannoWeierstrass property (BWP): A space X is said to have Bolzanno weierstrass property(BWP) if every finite set in X has a limit point.

NOTE:

A space with BWP is also said to be Frechet compact.

Countably compact spaces: A metric space X is said to be countably compact if every countable open cover of X has a finite subcover.

Sequentially compact spaces: A metric space (X, d) is known as sequentially compact if every sequence on X has a convergent subsequence.

- A metric space X is sequentially compact iff it has a B.W.P.

ε –net: Let (X, d) be a metric space and ε be an arbitrary positive number. Then a subset $A \subseteq X$ is said to be an ε −net for X,for any given $x \in X$, there exists a point $y \in A$ such that $d(x, y) < \varepsilon$, i.e., A is an ε −net for X if $X = \cup \{S(y, \varepsilon): y \in A\}$.

Finite ε –net:A finite subset of X that is an e-net for X is called a finite ε −net for X.

Lebesgue number for covers: Let (X, d) be a metric space and let $G = \{G_\lambda: \lambda \in \Lambda\}$ be an open cover of X. A real number $l > 0$ is said to be lebesgue number for G iff every subset of X with diameter less than l is contained in atleast one of G_λ.

- Every open cover of sequentially compact metric space has a lebesgue number.

Totally bounded: The metric space (X, d) is said to be totally bounded if, for any$\boldsymbol{\varepsilon} > 0$, there exists a finite $\boldsymbol{\varepsilon}$ −net for (X, d).

A nonempty subset Y of X is said to betotally bounded if the subspace Y is totally bounded.

Example:

A bounded interval in $\mathbb{R}$ is a totally bounded metric space. Let the endpoints of the interval be a and b $(a < b)$ and ε be an arbitrary positive number. Take an integer $n > \frac{b-a}{\varepsilon}$ and divide the interval into n equal subintervals each of length $\frac{b-a}{n}$.

The points

$\left\{a + \frac{(k-1)(b-a)}{n} : k = 2, \ldots, n\right\}$contain the required ε-net for the interval with endpoints a and b.

Let $x$be any point in the interval. Then $a \le x \le b$.

Then there exists an integer $\lambda \in \{1, 2, \ldots, n\}$ such that

$$a + \frac{(\lambda - 1)(b - a)}{n} \le x \le a \le a + \frac{\lambda(b - a)}{n}$$

Accordingly, the distance of x from each of the endpoints of the interval

$$\left[a + \frac{(\lambda - 1)(b - a)}{n}, a + \frac{\lambda(b - a)}{n}\right]$$

is less than or equal to $\frac{b-a}{n}$, which is strictly less than ε in view of the way in which n has been selected.

⇒any set containing at least one endpoint of each of the preceding subintervals, $k = 1, 2, \ldots, n$, forms an ε –net, the collection of points constitute the set.

- Every sequentially compact metric space (X, d) is totally bounded.
- A metric space X is compact if and only if it is sequentially compact.
- Every compact metric space is complete.
- A metric space is compact if and only if it is complete and totally bounded.

5.3.1 LEMMA

A compact subset M of a metric space is closed and bounded.

Proof. For every $x \in \overline{M}$ there is a sequence $\langle x_n \rangle$ in M such that

$$x_n \to x;$$

Since M is compact, $x \in M$.

Hence M is closed because $x \in \overline{M}$ was arbitrary.

We prove that M is bounded.

If M were unbounded, it would contain an unbounded sequence $\langle y_n \rangle$ such that,

$d(y_n, b) > n$, where b is any fixed element.

This sequence could not have a convergent subsequence since a convergent subsequence must be bounded [using previous result of metric space].

- The converse of this lemma is general false.

  Consider the sequence $< e_n >$ in l^2, where $e_n = (\delta_{nj})$ has the n^{th} term and all other terms 0;

  (e_n) is called a Schauder basis (or basis) for X.

$$e_1 = (1, 0, 0, 0 \ldots .)$$

$$e_2 = (0, 1, 0, 0 \ldots .)$$

$$e_3 = (0, 0, 1, 0 \ldots .)$$

  This sequence M is bounded since $\|e_n\| = 1$.

  It's terms constitute a point set which is closed because it has no point of accumulation. Therefore, M is not compact because we cannot produce a convergent subsequence of M.

5.3.2 THEOREMS

Theorem 3. In a finite dimensional normed space X, any subset $M \subset X$ is compact if and if M is closed and bounded.

Proof. Since compact subset M of a metric space is closed and bounded.

Let M be closed and bounded.

Let $\dim X = n$ and $\{e_1, e_2, \ldots \ldots . e_n\}$ a basis for X.

Consider any sequence $\{x_m\}$ in M.

Then each $x_m = \xi_1{}^{(m)} e_1 + \ \dots\dots\dots \xi_n{}^{(m)} e_n$.

Since M is bounded, so is $\{x_m\}$.

$\|x_m\| \leq k$ for all m.

By lemma in previous unit,

$k \geq \|x_m\| = \left\|\sum_{j=1}^{n} \xi_j^{(m)} e_j\right\| \geq c \sum_{j=1}^{n} \xi_j^{(m)}$.

Where $c > 0$.

Hence the sequence of numbers $\xi_j^{(m)} (j\ fixed)$ is bounded and by the Bolzano-Weierstrass theorem, has a point of accumulation ξ_j; here $1 \leq j \leq n$. Now, we can conclude that $\{x_m\}$ has a subsequence $\{z_m\}$ which converges $z = \sum \xi_j\, e_j$.(using lemma of previous unit).

Since M is closed, $z \in M$. This shows that the arbitrary sequence $\{x_m\}$ in M has a subsequence which converges in M. Hence M is compact.

5.4 F. RIESZ'S LEMMA

Let Y and Z be subspaces of a normed space X(of any dimension), and suppose that Y is closed and is a proper subset of Z. Then for every real number θ in the interval $(0,1)$ there is a $z \in Z$ such that,

$$\|z\| = 1, \|z - y\| \geq \theta \text{ for all } y \in Y.$$

Proof. Consider any $v \in Z - Y$ and denote its distance from Y by a.

$$a = \lim_{y \in Y} \|v - y\|.$$

Since Y is closed then $a > 0$. We now take any $\theta \in (0,1)$.

By the definition of an infimum there is a $y_0 \in Y$ such that,

$a \leqq \|v - y_0\| \leqq \frac{a}{\theta}$...........(1)

(Note that $\frac{a}{\theta} > a$ since $0 < \theta < 1$).

Let, $z = c(v - y_0)$ where $c = \frac{1}{\|v-y_0\|}$.

We have,

$$\|z - y\| = \|c(v - y_0) - y\|$$
$$= c\|v - y_0 - c^{-1}y\|$$
$$= c\|v - y_1\|$$

Then, $\|z\| = 1$, and we show that $\|z - y\| \geq \theta$ for every $y \in Y$.

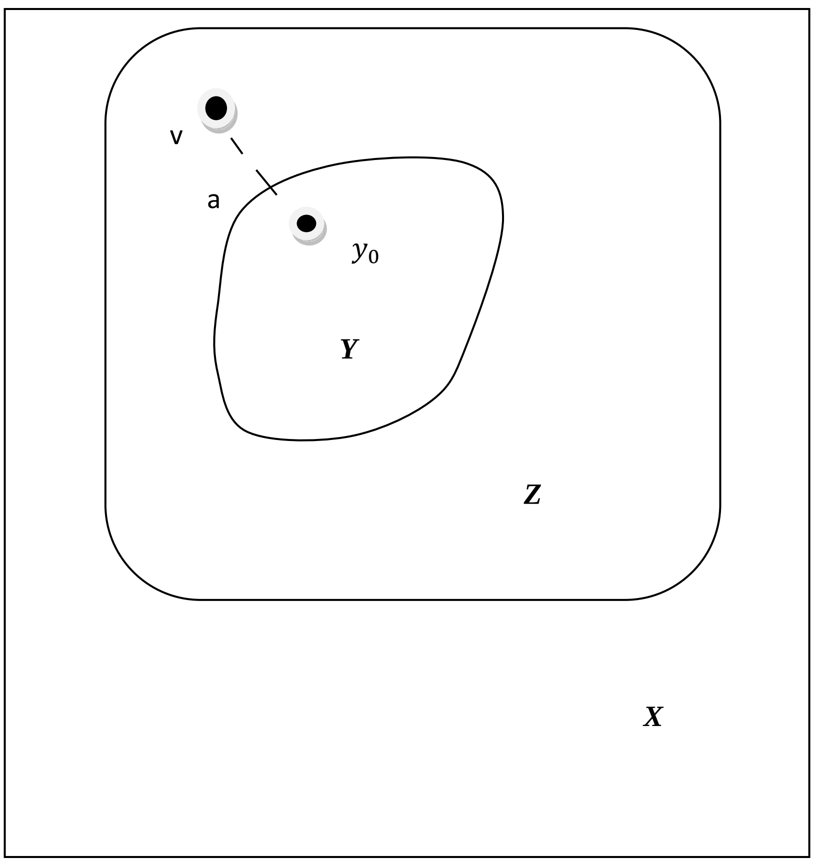

Fig.5.4.1

Hence $\|v - y_1\| \geq a$, by the definition of a.

Writing c out and using (1), we obtain

$$\|z - y\| = c\|v - y_1\| \geq ca = \frac{a}{\|v - y_0\|} \geq \frac{a}{a/\theta} = \theta.$$

Since $y \in Y$ was arbitrary, this completes the proof.

5.5 THEOREMS

Theorem 4. If a normed space X has the property that the closed unit ball $M = \{x | \|x\| \leqq 1\}$ is compact, then X is finite dimensional.

Proof. For proving this theorem assume that M is compact but

$$\dim X = \infty.$$

We choose any x_1 of norm 1.

This generates a one dimensional subspace X_1 of X, which is closed and is a proper subspace of X.

Since $\dim X = \infty$.

By Riesz's lemma there is an $x_2 \in X$ of norm 1 such that

$$\|x_2 - x_1\| \geq \theta = \frac{1}{2}.$$

The elements x_1, x_2 generate a two dimensional proper closed space X_2 of X.

By Riesz's lemma there is an $x_3 \in X$ of norm 1 such that

$$\|x_3 - x_2\| \geq \theta = \frac{1}{2}.$$

In particular $\|x_3 - x_2\| \geq \frac{1}{2}$.

Proceeding by induction, we obtain a sequence $< x_n >$ of elements x_n Such that,

$\|x_m - x_n\| \geq \frac{1}{2}$ (where $m \neq n$).

It implies that $< x_n >$ cannot have a convergent subsequence.

This contradicts the compactness of M.

Hence our assumption $\dim X = \infty$ is false and $\dim X < \infty$.

Theorem 5 (Continuous mapping).

Let X and Y be metric spaces and $T: X \to Y$ a continuous mapping. Then the image of compact subset M of X under T is compact.

Proof. By the definition of compactness it suffices to show that every sequence $< y_n >$ in the image of $T(M) \subset Y$ contains a subsequence which converges in $T(M)$.

Since $y_n \in T(M)$.

We have $y_n = T(x_n)$ for some $x_n \in M$.

Since M is compact.

$< x_n >$ contains a subsequence $< x_{n_k} >$ which converges in M.

Because $T: X \to Y$ a continuous mapping.

The image of $< x_{n_k} >$ is a subsequence of $< y_n >$ which converges in $T(M)$.

Hence $T(M)$ is compact.

5.5.1 COROLLARY

A continuous mapping T of a compact subset M of a metric space X into $\mathbb{R}$ assumes a maximum and a minimum at some points of M.

Proof. Since we know that Let X and Y be metric spaces and $T: X \to Y$ a continuous mapping. Then the image of compact subset M of X under T is compact and a compact subset M of a metric space is closed and bounded.

So that,

$$inf T(M) \in T(M),$$
$$sup\, T(M) \in T(M).$$

And the inverse images of these two points consist of points of M at which Tx is minimum or maximum, respectively.

5.6 SUMMARY

Present unit is presentation of the concepts Compactness explained the topic of F. Riesz's Lemma. These concepts have been explained with the help of definitions, examples and theorems. The learners can understand the concepts in easy manner.

5.7 GLOSSARY

i. **Metric space:** Let $X \neq \emptyset$ be a set then the metric on the set X is defined as a function $d: X \times X \to [0, \infty)$ such that some conditions are satisfied.

ii. **Vector space: -** Let V be a nonempty set with two operations

   (i) **Vector addition:** If any $u, v \in V$ then $u + v \in V$

   (ii) **Scalar Multiplication:** If any $u \in V$ and $k \in F$ then $ku \in V$

   Then V is called a vector space (over the field F) if the following axioms hold for any vectors if the some conditions hold.

iii. **Normed space:-** Let X be a vector space over scalar field K. A *norm* on a (real or complex) vector space X is a real-valued function on X ($\|x\|: X \to K$) whose value at an $x \in X$ is denoted by $\|x\|$ and which has the four properties here x and y are arbitrary vectors in X and α is any scalar.

iv. **Banach space:-** A complete normed linear space is called a Banach space.

v. Quotient Spaces

vi. Subspace

vii. Finite dimensional Normed Spaces

viii. Equivalent norms

CHECK YOUR PROGRESS

1. Any bounded subset in $\mathbb{R}^n$ is :

a) compact

b) relatively compact

c) open

d) closed

2. Consider the statements:

(i) Every compact operator is bounded.

(ii) Every bounded operator is compact. Then:

(a) Only (i) is true.

(b) Only (ii) is true.

(c) Both (i) and (ii) are true.

(d) Neither (i) nor (ii) is true.

3. A metric space X is said to be compact if every sequence in X has a convergent subsequence. True/False.

4. A compact subset M of a metric space is not closed and bounded. True/False.

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5.9 SUGGESTED READINGS

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5.10 TERMINAL QUESTIONS

1. What is a finite dimensional?

   ………………………………………………………………

2. What is compactness in Banach space?

   ………………………………………………………………

3. What is an example of a finite-dimensional?
…………………………………………………………………
4. State and Prove is the Riesz Lemma?
………………………………………………………………..

5.11 ANSWERS

CHECK YOUR PROGRESS

1) b.
2) a.
3) True.
4) False.

BLOCK II: LINEAR FUNCTIONAL AND LINEAR OPERATOR

UNIT 6:

LINEAR OPERATORS

CONTENTS:

6.1 INTRODUCTION

In previous units we have studied about Normed Space, Banach Space, Finite dimensional Spaces and Compactness and Finite Dimension. In present unit is a presentation of concepts of linear operator. An operator in mathematics is typically a mapping or function that modifies one space's components to create another space's elements. Although the term "operator" has no universal definition, it is frequently used in place of "function" when the domain is a collection of functions or other organized objects. Furthermore, it might be challenging to clearly define an operator's domain because it can be expanded to function on related objects. Acting on vector spaces, linear maps are the most fundamental operators. Linear operators are linear maps with the same space, for example from $\mathbb{R}^{\boldsymbol{n}}$ to $\mathbb{R}^{\mathbf{n}}$ serving as both the domain and the range. These operators frequently maintain characteristics like continuity.

We are using the following notations:

i. $\mathcal{D}(T) =$ domain of T.

ii. $\mathcal{R}(T) =$ range of T.

iii. $\mathcal{N}(T) =$ null space of T.

In this unit we shall elaborate somewhat on the theory of operators. In so doing, we will define several important types of operators, and we will also prove some important theorems.

6.2 OBJECTIVES

After studying this unit, learner will be able to

- **i.** Described the concept of Linear operator
- **ii.** Explained the topic of Bounded and Continuous Linear Operators.
- **iii.** Defined the concept of Integral operator.

6.3 LINEAR OPERATOR

A *linear operator* T is an operator such that

i. The domain $\mathcal{D}(T)$ of T is a vector space and the range $\mathcal{R}(T)$ lies in a vector space over the same field.

ii. for all $x, y \in \mathcal{D}(T)$ and scalar α,

$$T(x+y) = Tx + Ty,$$
$$T(\alpha x) = \alpha Tx.$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

By definition, the null space of T is the set of all $x \in \mathcal{D}(T)$ such that $Tx = 0$.

- We can use another word for null space is "kernel".
- Equation (1) shows that the linear operator 'T' is a homomorphism from one vector space to another vector space, that is T save two operations on the vector space.

Range Space: The range space of an operator $T : X \to Y$, denoted $\mathscr{R}(T)$, is the set of all vectors $y_i \in Y$ such that for every $y_i \in \mathscr{R}(T)$ there exists an x $\in$X such that $Tx = y_i$.

Null Space: The null space of operator T, denoted $\mathcal{N}(T)$ is the set of all vectors $x_i \in X$ such that $T(x_i) = 0$:

$$\mathcal{N}(T) = \{x_i \in X \mid Tx_i = 0\}.$$

- For a linear operator T, the Null Space $\mathcal{N}(T)$ is a subspace of X. Furthermore, if T is continuous (in a normed space X), then $\mathcal{N}(T)$is closed
- A linear operator on a normed space X (to a normed space Y) is continuous at every point X if it is continuous at a single point in X

Definition: A *linear transformation* is a function T from U into V (where $U(F)$ and $V(F)$ be two vector spaces) such that for all α, β in U and for all $a, b \in F$. $T(\alpha + \beta) = T(\alpha) + T(\beta), T(a\alpha) = aT\alpha$.

Or $\quad T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$.

A *linear operator* T is a mapping from same vector space to same vector space. The field will be same in vector space. *Linear transformation* T is a mapping from one vector space to another vector space. The field are same in both the cases.

6.3.1 EXAMPLES

Example 1: The identity operator $I_X: X \to X$ is defined by $I_X x = x$ for all $x \in X$. If write I in place of I_X. Thus, $Ix = x$.

Example 2: The zero operator $0: X \to Y$ is defined by $0x = x$ for all $x \in X$.

Example 3: Let X be the vector space of all polynomial on $[a, b]$. We may define a linear operator T on X by setting,

$$Tx(t) = x'(t)$$

for every $x \in X$, where the prime denotes differentiation with respect to t. This operator T maps X onto itself.

Example 4: A linear operator T from $C[a, b]$ into itself can be defined as $Tx(t) = \int_0^t x(\tau)\, d\tau$ and also $Tx(t) = tx(t)$ where $t \in [a, b]$.

Example 5: A real matrix $A = (\alpha_{jk})$ with r rows and n columns defines an operator $T: \mathbb{R}^n \to \mathbb{R}^r$ by means of $y = Ax$ where $x = (\xi_j)$ has n components and $y = (\eta_i)$ has r components and both vectors are written as column vectors because of the usual convention of matrix multiplication; writing $y = Ax$ out, we have

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ . \\ \eta_r \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots \alpha_{2n} \\ . & . & \dots \ . \\ . & . & \dots \ . \\ \alpha_{r1} & \alpha_{r2} & \dots \alpha_{rn} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ . \\ \xi_n \end{bmatrix}$$

T

If A were complex, it would define a linear operator operator from $\mathbb{C}^n$ into $\mathbb{C}^r$.

Note: In above example the dimension of domain is n and dimension of range is r.

Example 6: The function $T:V_3(R)\to V_2(R)$

Defined by $T(a,b,c)=(a,b)\forall a,b\in R$ is a linear transformation from $V_3(R)$ into $V_2(R)$.

Let $\alpha=(a_1,b_1,c_1),\beta=(a_2,b_2,c_2)\in V_3(R)$

If $a,b\in R$, then

$T(a\alpha+b\beta)=T[a(a_1,b_1,c_1)+b(a_2,b_2,c_2)]$

$=T(aa_1+ba_2,ab_1+bb_2,cc_1+bc_2)$

$=(aa_1+ba_2,ab_1+bb_2)$ [by def. of T]

$=(aa_1,ab_1)+(ba_2,bb_2)$

$=a(a_1,b_1)+b(a_2,b_2)$

$=a(a_1,b_1,c_1)+b(a_2,b_2,c_2)$

$=aT(\alpha)+bT(\beta)$.

Hence T is a linear transformation from $V_3(R)$ into $V_2(R)$

Example 7: Show that the mapping $T:V_3(R)\to V_2(R)$ defined as $T(a_1,a_2,a_3)=(3a_1-2a_2+a_3,a_1-3a_2-2a_3)$ is a linear transformation from $V_3(R)$ to $V_2(R)$

Proof: Let $\alpha=(a_1,a_2,a_3),\beta=(b_1,b_2,b_3)\in V_3(R)$.
Then $T(\alpha)=T(a_1,a_2,a_3)=(3a_1-2a_2+a_3,a_1-3a_2-2a_3)$
And $T(\beta)=T(b_1,b_2,b_3)=(3b_1-2b_2+b_3,b_1-3b_2-2b_3)$.
Let $a,b\in R$. Then $a\alpha+b\beta\in V_3(R)$. We have
$T(a\alpha+b\beta)=T[a(a_1,a_2,a_3)+b(b_1,b_2,b_3)]$
$=T(aa_1+bb_1,aa_2+bb_2,aa_3+bb_3)$

$= (3(3aa_1 + bb_1) - 2(aa_2 + bb_2) + aa_3 + bb_3, aa_1 + bb_1 - 3(aa_2 + bb_2) - 2(aa_3 + bb_3))$

$= (a(3a_1 - 2a_2 + a_3) + b(3b_1 - 2b_2 + b_3), a(a_1 - 3a_2 - 2a_3) + b(b_1 - 3b_2 - 2b_3)$
$= a(3a_1 - 2a_2 + a_3, a_1 - 3a_2 - 2a_3) + b(3b_1 - 2b_2 + b_3, b_1 - 3b_2 - 2b_3)$
$= aT(\alpha) + bT(\beta)$

Example 8: Show that the mapping $T : V_2(R) \to V_3(R)$ defined as

$$T(a,b) = (a+b, a-b, b)$$

is a linear transformation from $V_2(R)$ into $V_3(R)$.

Solution: Let the vectors $\alpha = (a_1, b_1), \beta = (a_2, b_2) \in V_2(R)$.
Then $T(\alpha) = T(a_1, b_1) = (a_1 + b_1, a_1 - b_1, b_1)$ and
$T(\beta) = (a_2 + b_2, a_2 - b_2, b_2)$.
Also let $a, b \in R$. Then $a\alpha + b\beta \in V_2(R)$ and
$T(a\alpha + b\beta) = T[a(a_1, b_1) + b(a_2, b_2)]$
$= T(aa_1 + ba_2, ab_1 + bb_2)$
$= (aa_1 + ba_2 + ab_1 + bb_2, aa_1 + ba_2 - ab_1 - bb_2, ab_1 + bb_2)$
$= a(a_1 + b_1, a_1 - b_1, b_1) + b(a_2 + b_2, a_2 - b_2, b_2)$
$= aT(\alpha) + bT(\beta)$
$\therefore T$ is a linear transformation from $V_2(R)$ into $V_3(R)$.

6.4 THEOREM

Theorem 1: Let T be a linear operator. Then:

- **a.** The range $\mathcal{R}(T)$ is a vector space.
- **b.** If $\dim \mathcal{D}(T) = n < \infty$, then $\dim \mathcal{R}(T) \leqq n$.
- **c.** The null space $\mathcal{N}(T)$ is a vector space.

Proof. a. We take $y_1, y_2 \in \mathcal{R}(T)$.

We have to show that $\alpha y_1 + \beta y_2 \in \mathcal{R}(T)$ for any scalars α, β.

Since $y_1, y_2 \in \mathcal{R}(T)$.

We have $y_1 = Tx_1,\ y_2 = Tx_2,$

for some $x_1, x_2 \in \mathcal{D}(T)$ and $\alpha x_1 + \beta x_2 \in \mathcal{D}(T)$.

Because $\mathcal{D}(T)$ is a vector space.

The linearity of T gives,

$$T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2 = \alpha y_1 + \beta y_2.$$

Hence $\alpha y_1 + \beta y_2 \in \mathcal{R}(T)$.

Since $y_1, y_2 \in \mathcal{R}(T)$ were arbitrary and α, β are the any scalars.

This proves that $\mathcal{R}(T)$ is a vector space.

b. We choose $n+1$ elements $y_1, y_2, \dots \dots \dots y_{n+1}$ of $\mathcal{R}(T)$ in arbitrary manner.

Then, we have $y_1 = Tx_1,\ y_2 = Tx_2, \dots \dots,\ y_{n+1} = Tx_{n+1}$ for some $x_1, \dots \dots \dots., x_{n+1}$ in $\mathcal{D}(T)$.

Since dim $\mathcal{D}(T) = n$.

The set $\{x_1, \dots \dots \dots., x_{n+1}\}$ must be linearly dependent.

Hence,

$\alpha_1 x_1 + \dots \dots \dots \dots \dots \dots \dots \dots \dots. + \alpha_{n+1} x_{n+1} = 0.$

For some scalars $\alpha_1, \dots \dots .., \alpha_{n+1,}$ not all zero.

Since T is linear and $T(0) = 0$.

Application of T on both sides gives,

$$T(\alpha_1 x_1 + \dots \dots \dots \dots \dots \dots \dots \dots \dots. + \alpha_{n+1} x_{n+1})$$

$$= \alpha_1 y_1 + \dots \dots \dots \dots \dots \dots \dots \dots \dots. + \alpha_{n+1} y_{n+1} = 0.$$

This shows that $\{y_1, y_2, \dots \dots \dots y_{n+1}\}$ is a linearly dependent set because a_i's are not all zero.

Since this subset of $\mathcal{R}(T)$ was chosen in arbitrary manner.

We conclude that $\mathcal{R}(T)$ has no linearly independent subsets of $n+1$ or more elements.

By the definition this means that dim $\mathcal{R}(T) \leqq n$.

c. We take any $x_1, x_2 \in \mathcal{N}(T)$.

Then $Tx_1 = Tx_2 = 0$.

Since T is linear, for any scalars α, β, we have

$$T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2 = 0.$$

This shows that $\alpha T x_1 + \beta T x_2 \in \mathcal{N}(T)$.

The null space $\mathcal{N}(T)$ is a vector space.

- Second part meaning that linear operator preserve linear dependence.

6.5 INVERSE OF T

A mapping $T: \mathcal{D}(T) \to Y$ is said to be injective or one –to-one if different points in the domain have different images, that is, if for any $x_1, x_2 \in \mathcal{D}(T)$,

$$x_1 \neq x_2 \Rightarrow Tx_1 \neq Tx_2; \dots\dots\dots\dots (2)$$

It is also equivalent,

$$Tx_1 = Tx_2 \Rightarrow x_1 = x_2 \quad \dots\dots\dots\dots\dots(3)$$

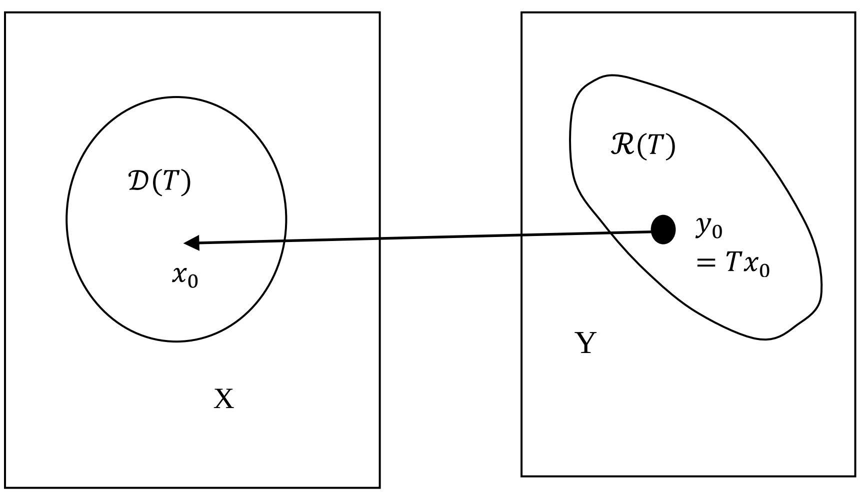


Fig.6.5.1

In this case there exists the mapping

$$T^{-1}: \mathcal{R}(T) \to \mathcal{D}(T),$$

$$y_0 \mapsto x_0 \ (y_0 = Tx_0)$$

……………………………………..(4)

which maps every $y_0 \in \mathcal{R}(T)$ onto that $x_0 \in \mathcal{D}(T)$ for which

$Tx_0 = y_0$.

The mapping T^{-1} is called the inverse of T.

From (4) it is clear that,

$T^{-1}Tx = x$ for all $x \in \mathcal{D}(T)$.

$TT^{-1}y = y$ for all $y \in \mathcal{R}(T)$.

Note: The inverse of a linear operator exists if and only if the null space of the operator consists of the zero vector only.

6.6 THEOREM AND LEMMA

Theorem 2: Let X and Y be vector spaces both real or complex. Let $T: \mathcal{D}(T) \to Y$ be a linear operator with domain $\mathcal{D}(T) \subset X$ and $\mathcal{R}(T) \subset Y$. Then:

a) The inverse $T^{-1}: \mathcal{R}(T) \to \mathcal{D}(T)$ exists if and only if

$$Tx = x \Longrightarrow x = 0.$$

b) If T^{-1} exists, it is a linear operator.

c) If $\dim \mathcal{D}(T) = n < \infty$ and T^{-1} exists, then $\dim \mathcal{R}(T) = \dim \mathcal{D}(T)$.

Proof. **a)** Suppose that $Tx = 0$ implies $x = 0$. Let $Tx_1 = Tx_2$. Since T is linear,

$$T(x_1 - x_2) = Tx_1 - Tx_2 = 0,$$

so that $x_1 - x_2 = 0$ by the hypothesis. Hence $Tx_1 = Tx_2$ implies $x_1 = x_2$ and T^{-1} exists, then (3) satisfy. From (3) with $x_2 = 0$.

Since $T0 = 0$. We obtain,

$$Tx_1 = T0 = 0 \Rightarrow x_1 = 0.$$

This completes the proof of a).

b) We consider that T^{-1} exists and show that T^{-1} is linear.

The domain of T^{-1} is $\mathcal{R}(T)$ and is a vector space by Theorem 1(a).

We are assuming for any $x_1, x_2 \in \mathcal{D}(T)$ and their images

$y_1 = Tx_1$ and $y_2 = Tx_2$.

Then $x_1 = T^{-1}y_1$ and $x_2 = T^{-1}y_2$.

T is linear, so that for any scalars α and β we have,

$\alpha y_1 + \beta y_2 = \alpha Tx_1 + \beta Tx_2 = T(\alpha x_1 + \beta x_2)$.

Since $x_i = T^{-1}y_i$, this implies

$T^{-1}(\alpha y_1 + \beta y_2) = \alpha x_1 + \beta x_2 = \alpha T^{-1}y_1 + \beta T^{-1}y_2$

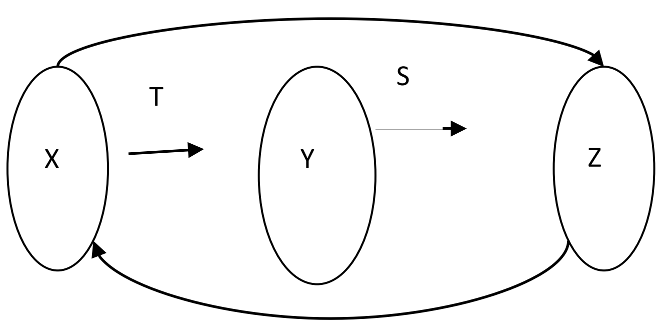
and proves that T^{-1} is linear.

c) We have $\dim \mathcal{R}(T) \leqq \dim \mathcal{D}(T)$ by Theorem 1(b), and $\dim \mathcal{D}(T) \leqq \dim \mathcal{R}(T)$ by the same theorem applied to T^{-1}.

Inverse of the composite of linear operators:

Lemma 1: Let $T: X \to Y$ and $S: Y \to Z$ be bijective linear operators, where X, Y, Z are vector spaces. Then the $(ST)^{-1}: Z \longrightarrow X$ of the product (the composite) ST exists, and $(ST)^{-1} = T^{-1}S^{-1}$.

ST



$(ST)^{-1}$

Fig 6.6.1

Proof. The operator $ST: X \to Z$ is bijective, so that $(ST)^{-1}$exists.

$$ST(ST)^{-1} = I_Z$$

where I_Z is the identity operator on Z.

Applying S^{-1} and using $S^{-1}S = I_Y$ (the identity operator on Y), we obtain,

$$S^{-1}ST(ST)^{-1} = T(ST)^{-1} = S^{-1}I_Z = S^{-1}.$$

Applying T^{-1} and using $T^{-1}T = I_X$, we obtain the desired result

$$T^{-1}T(ST)^{-1} = (ST)^{-1} = T^{-1}S^{-1}.$$

This completes the proof.

Check Your Progress

1. Which of the following is true?

(a) If A, B are invertible linear operators on X, then $A + B$ is invertible.

(b) If A, B are invertible linear operators on X, then $A - B$ is invertible.

(c) If A, B are invertible linear operators on X, then AB is invertible.

(d) If A is invertible linear operator on X, and k is any scalar, then kA is invertible.

6.7 BOUNDED AND CONTINUOUS LINEAR OPERATORS

Let X and Y be normed spaces. Let $T: \mathcal{D}(T) \to Y$ be a linear operator with domain $\mathcal{D}(T) \subset X$. The operator T is said to be bounded if there is a real number c such that for all $x \in \mathcal{D}(T)$,

$$\|Tx\| \leq c\|x\|. \text{ ....(a)}$$

The value of c must be at least as big as the supremum of the expression on the left taken over $\mathcal{D}(T) - \{0\}$. Let $T: \mathcal{D}(T) - \{0\} \to Y$ be a linear operator with domain $\mathcal{D}(T) - \{0\} \subset X$ then, $\|T\| = \sup\limits_{\substack{x \in \mathcal{D}(T) \\ x \neq 0}} \frac{\|Tx\|}{\|x\|}$.

If $\mathcal{D}(T) = \{0\}$. Then $\|T\| = 0$. In this case $T = 0$. The operator T is said to be bounded if there is a real number c such that for all $x \in \mathcal{D}(T)$. When $c = \|T\|$ is

$$\|Tx\| \leq \|T\|\|x\| \text{ ....(b)}$$

Let $T: \mathcal{D}(T) \to Y$ be a bounded linear operator then ,

$$\|T\| = \sup_{\substack{x \in \mathcal{D}(T) \\ \|x\|=1}} \|Tx\| \text{ ....(c)}$$

Example 1: The identity operator $I_X: X \to X$ is defined by $I_X x = x$ for all $x \in X$. If write I in place of I_X. Thus, $Ix = x$.

Example 2: The zero operator $0: X \to Y$ is defined by $0x = x$ for all $x \in X$.

6.8 CONTINUITY OF OPERATOR

Let $T: \mathcal{D}(T) \to Y$ be any operator not necessarily linear, where $\mathcal{D}(T) \subset X$ and X and Y are normed spaces.

The operator T is continuous at an $x_0 \in \mathcal{D}(T)$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that,

$\|Tx - Tx_0\| < \varepsilon$ for all $x \in \mathcal{D}(T)$satisfying $\|x - x_0\| < \delta$.

T is continuous if T is continuous at every $x \in \mathcal{D}(T)$.

6.9 THEOREMS

Theorem 3. If a normed space X is finite dimensional, then every linear operator on X is bounded.

Proof. Let dim $X = n$ and $\{e_1, \dots \dots e_n\}$ a basis for X.

We take $x = \sum \xi_j e_j$ and consider any linear operator T on X.

Since, T is linear,

$$\|Tx\| = \left\|\sum \xi_j T e_j\right\|$$
$$\leq \sum |\xi_j| \, \|Te_j\|$$
$$\leq \max_k \|Te_j\| \sum |\xi_j|$$

(summations from 1 to n).

To the last sum we are applying the following result:

Let $\{x_1, \dots x_n\}$ be a linearly independent set of vectors in a normed space X $(of\ any\ dimension)$, then there is a number $c > 0$ such that for every choice of scalar $\alpha_1, \dots \dots \dots \alpha_n$ we have,

$\|\alpha_1 x_1 + \cdots \dots \dots \dots \dots + \alpha_n x_n\| \geqq c(|\alpha_1| + \cdots |\alpha_n|)$(from unit 4).

$\alpha_j = \xi_j$ and $x_j = e_j$. Then we obtain,

$$\sum |\xi_j| \leq \frac{1}{c} \left\|\sum \xi_j T e_j\right\| = \frac{1}{c} \|x\|.$$

Together,

$\|Tx\| \leq \gamma \|x\|$ where $\gamma = \frac{1}{c} \max_k \|Te_k\|$

Since $\|Tx\| \leq c\|x\|$.

Using the above condition we can say that T is bounded.

Theorem 4. Let $T: \mathcal{D}(T) \to Y$ be a linear operator, where $\mathcal{D}(T) \subset X$ and X and Y are normed spaces. Then:

i. T is continuous if and only if T is bounded.

ii. If T is continuous at a single point, it is continuous.

Proof.

i. For $T = 0$.

The statement is trivial.

Let $T \neq 0$.

Then, $\|T\| \neq 0$.

We assume T to be bounded and consider any $x_0 \in \mathcal{D}(T)$.

Let any $\varepsilon > 0$ be given.

Then, since T is linear, for every $x \in \mathcal{D}(T)$ such that,

$\|x - x_0\| < \delta$ where $\delta = \frac{\varepsilon}{\|T\|}$.

We obtain $\|Tx - Tx_0\| = \|T(x - x_0)\|$

$$\leq \|T\|\|x - x_0\|$$

$$< \|T\|\delta = \varepsilon.$$

Since, $x_0 \in \mathcal{D}(T)$ was arbitrary, this shows that T is continuous.

Conversely, assume that T is continuous at an arbitrary $x_0 \in \mathcal{D}(T)$.

Then, given any $\varepsilon > 0$, there is a $\delta > 0$ such that,

$\|Tx - Tx_0\| \leq \varepsilon$ for all $x \in \mathcal{D}(T)$

satisfying $\|x - x_0\| \leq \delta$.................................(d)

We now take any $y \neq 0$ in $\mathcal{D}(T)$ and set

$$x = x_0 + \frac{\delta}{\|y\|}y.$$

Then $x - x_0 = \frac{\delta}{\|y\|}y$.

Hence, $\|x - x_0\| = \delta$, so that we are using (d).

Since T is linear, we have

$$\|Tx - Tx_0\| = \left\|T\left(\frac{\delta}{\|y\|}\right)y\right\| = \frac{\delta}{\|y\|}\|Ty\| .$$

And (d) implies,

$$\frac{\delta}{\|y\|}\|Ty\| \leq \varepsilon.$$

Thus, $\left\|T\left(\frac{\delta}{\|y\|}\right)y\right\| = \frac{\varepsilon}{\delta}\|Ty\|$.

This implies $\|Ty\| \leq c\,\|Ty\|$. Where $c = \frac{\varepsilon}{\delta}$, and shows that T is bounded.

ii. Continuity of T at a point implies boundedness of T by the second part of the proof of (i), which is turn implies continuous of T by (i).

Corollary: Let T be a bounded linear operator. Then:

(a) *$x_n \longrightarrow x$ [where $x_n, x \in \mathfrak{D}(T)$] implies $Tx_n \longrightarrow Tx$.*

(b) *The null space $\mathcal{N}(T)$ is closed.*

Check Your Progress

2.

If X and Y are normed spaces, and if $T : X \to Y$ is a linear operator, then T is bounded if and only if:

(a) T maps bounded subsets of X into bounded subsets of Y.

(b) T maps open subsets of X into open subsets of Y.

(c) T maps closed subsets of X into closed subsets of Y.

(d) T is invertible.

3.

For any bounded linear operator $A : X \to Y$, $\ker A$ is:

(a) a closed subspace of Y.

(b) an open subspace of Y.

(c) a closed subspace of X.

(d) an open subspace of X.

Equal operator:

Two operators T_1 and T_2 are defined to be **equal**, written

$$T_1 = T_2,$$

if they have the same domain $\mathfrak{D}(T_1) = \mathfrak{D}(T_2)$ and if $T_1x = T_2x$ for all $x \in \mathfrak{D}(T_1) = \mathfrak{D}(T_2)$.

Restriction of an operator:

The **restriction** of an operator $T\colon \mathfrak{D}(T) \longrightarrow Y$ to a subset $B \subset \mathfrak{D}(T)$ is denoted by

$$T|_B$$

and is the operator defined by

$$T|_B\colon B \longrightarrow Y, \qquad T|_B x = Tx \text{ for all } x \in B.$$

Extension in operator:

An **extension** of T to a set $M \supset \mathfrak{D}(T)$ is an operator

$$\tilde{T}\colon M \longrightarrow Y \qquad \text{such that} \qquad \tilde{T}|_{\mathfrak{D}(T)} = T,$$

that is, $\tilde{T}x = Tx$ for all $x \in \mathfrak{D}(T)$.

[Hence T is the restriction of $\tilde{T}$ to $\mathfrak{D}(T)$.]

Theorem (Bounded linear extension). Let

$$T\colon \mathfrak{D}(T) \longrightarrow Y$$

be a bounded linear operator, where $\mathfrak{D}(T)$ lies in a normed space X and Y is a Banach space. Then T has an extension

$$\tilde{T}\colon \overline{\mathfrak{D}(T)} \longrightarrow Y$$

where $\tilde{T}$ is a bounded linear operator of norm

$$\|\tilde{T}\| = \|T\|.$$

6.10 SUMMARY

In this unit we explain the concept of Linear operator: Definition, Examples, Theorem and Inverse of T: Theorem and Lemma. We also present the concept of Bounded and Continuous Linear Operator and Continuity of Linear Operator. At the end of the unit learner will be able to understand the basic concepts of operator theory.

6.11 GLOSSARY

i. **Metric space:** Let $X \neq \emptyset$ be a set then the metric on the set X is defined as a function $d: X \times X \to [0, \infty)$ such that some conditions are satisfied.

ii. **Vector space: -** Let V be a nonempty set with two operations

(i) **Vector addition:** If any $u, v \in V$ then $u + v \in V$

(ii) **Scalar Multiplication:** If any $u \in V$ and $k \in F$ then $ku \in V$

Then V is called a vector space (over the field F) if the following axioms hold for any vectors if the some conditions hold.

iii. **Normed space:-** Let X be a vector space over scalar field K. A *norm* on a (real or complex) vector space X is a real-valued function on X ($\|x\|: X \to K$) whose value at an $x \in X$ is denoted by $\|x\|$ and which has the four properties here x and y are arbitrary vectors in X and α is any scalar.

iv. **Banach space:-** A complete normed linear space is called a Banach space.

v. Quotient Spaces

vi. Subspace

vii. Finite dimensional Normed Spaces

viii. Equivalent norms

CHECK YOUR PROGRESS

4.

Consider the statements:

(i) Every compact operator is bounded.

(ii) Every bounded operator is compact. Then:

(a) Only (i) is true.

(b) Only (ii) is true.

(c) Both (i) and (ii) are true.

(d) Neither (i) nor (ii) is true.

5.

Every bounded operator of finite rank is :

(a) compact

(b) open

(c) has a non zero adjoint.

(d) None of these.

6.

Rank of a linear operator A equals:

(a) $\dim(\ker A)$

(b) $\dim(\text{Im} A)$

(c) $\dim(\text{Im} A^*)$

(d) $\dim(\ker A^*)$

7.

If T is a bounded linear operator, then:

(a) $\|Tx\| \leq \|T\| \cdot \|x\|$

(b) $\|Tx\| \geq \|T\| \cdot \|x\|$

(c) $\|Tx\| = \|T\| \cdot \|x\|$

(d) None of these.

8.

Let E be a normed space and A, B be bounded linear operators on E. Then which of the following is true?

(a) $\|AB\| \leq \|A\| \cdot \|B\|$

(b) $\|AB\| \geq \|A\| \cdot \|B\|$

(c) $\|AB\| = \|A\| \cdot \|B\|$

(d) None of these.

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6.12 *SUGGESTED READINGS*

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ii. J. B. Conway, (1990). *A Course in functional Analysis* (4^{th} Edition), Springer.

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6.13 TERMINAL QUESTIONS

1.

Let X and Y be normed spaces. Show that a linear operator $T: X \longrightarrow Y$ is bounded if and only if T maps bounded sets in X into bounded sets in Y.

2.

If $T \neq 0$ is a bounded linear operator, show that for any $x \in \mathfrak{D}(T)$ such that $\|x\| < 1$ we have the strict inequality $\|Tx\| < \|T\|$.

3.

Defined Linear operator……………………………….

4.

Defined Bounded and Continuous Linear Operator…………

6.14 ANSWERS

CHECK YOUR PROGRESS

1. c
2. a
3. c
4. a
5. a
6. b
7. a
8. a

UNIT 7:

LINEAR FUNCTIONAL

CONTENTS:

7.1 INTRODUCTION

In previous unit we have studied about linear operator. In this unit we are studying about linear functional. The aim of this unit is to present the basic facts of linear functional analysis related to applications to some fundamental aspects of mathematical analysis. In functional analysis, individual functions satisfying specific equations are replaced by classes of functions and transforms which are determined by each particular problem. The objects of functional analysis are spaces and operators acting between them which, after systematic studies intertwining linear and topological or metric structures, appear to be behind classical problems in a kind of cleaning process.

In mathematics, a linear form (also known as a linear functional, a one-form, or a covector) is a linear map from a vector space to its field of scalars (often, the real numbers or the complex numbers).

A functional is an operator whose range lies on the real line $\mathbb{R}$ or in the complex plane $\mathbb{C}$. And functional analysis was initially the analysis of functionals. The latter appear so frequently that special notations are used. We denote functionals by lowercase letters $f, g, h, \dots\dots\dots$ the domain of f by $\mathcal{D}(f)$, the range of f by $\mathcal{R}(f)$ and the value of f at an $x \in \mathcal{D}(f)$ by $f(x)$ with parentheses. Functionals are operators, so that previous definitions apply.

7.2 OBJECTIVES

After studying this unit, learner will be able to

- **i.** Described the concept of Linear functional
- **ii.** Explained the topic of Bounded linear functional

iii. Proved the theorem and solve the examples based on linear functional.

7.3 LINEAR FUNCTIONAL

Definition:

A *linear functional f* is a linear operator with domain in a vector space X and range in the scalar field K of X; thus, $f\colon \mathcal{D}(f) \longrightarrow K,$

where $K = \mathbf{R}$ if X is real and $K = \mathbf{C}$ if X is complex.

7.4 *BOUNDED LINEAR FUNCTIONAL*

A bounded linear functional f is a bounded linear operator (definition in previous unit) with range in the scalar field of the normed space X in which the domain $\mathcal{D}(f)$ lies. Thus there exists a real number c such that for all $x \in \mathcal{D}(f)$.

$$|f(x)| \leqq c\,\|x\|. \qquad \text{……………….……….(1)}$$

In continuation norm is defined in a way

$$\|f\| = \sup_{\substack{x \in \mathcal{D}(f) \\ x \neq 0}} \frac{|f(x)|}{\|x\|} \qquad \text{……………..(2)}$$

Similarly,

$$\|f\| = \sup_{\substack{x \in \mathcal{D}(f) \\ \|x\| = 1}} |f(x)|. \qquad \text{……………………(3)}$$

$$|f(x)| \leqq \|f\| \, \|x\|, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Theorem : A linear functional, with domain $\mathcal{D}(f)$ in a normed space is continuous if and only if f is bounded.

7. 5 *EXAMPLES*

Example 1:

Norm. The *norm* $\|\cdot\|: X \longrightarrow \mathbf{R}$ on a normed space $(X, \|\cdot\|)$ is a functional on X which is not linear.

Example 2:

Dot product. The familiar *dot product* with one factor kept fixed defines a functional $f: \mathbf{R}^3 \longrightarrow \mathbf{R}$ by means of

$$f(x) = x \cdot a = \xi_1\alpha_1 + \xi_2\alpha_2 + \xi_3\alpha_3,$$

where $a = (\alpha_j) \in \mathbf{R}^3$ is fixed.

Example 3:

The constant zero function, mapping every vector to zero, is trivially a linear functional. Every other linear functional (such as the ones below) is surjective, (that is, its range is all of k).

Example 4:

The mean element of an n-vector is given by the one-form

$[1/n, 1/n, \ldots, 1/n]$. That is,

$$\text{mean}(v) = [1/n, 1/n, \ldots, 1/n] \cdot v.$$

Example 5:

Sampling with a kernel can be considered a one-form, where the one-form is the kernel shifted to the appropriate location.

Example 6:

Linear functionals in $\mathbb{R}^n$:

Suppose that vectors in the real coordinate space $\mathbb{R}^n$ are represented as column vectors

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

For each row vector $\mathbf{a} = [a_1 \quad \cdots \quad a_n]$,

there is a linear functional $f_{\mathbf{a}}$ defined by

$$f_{\mathbf{a}}(\mathbf{x}) = a_1 x_1 + \cdots + a_n x_n,$$

and each linear functional can be expressed in this form.

This can be interpreted as either the matrix product or the dot product of the row vector a and the column vector x:

$$f_{\mathbf{a}}(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x} = [a_1 \quad \cdots \quad a_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Example 7:

The definite integral is a number if we consider it for a single function, as we do in calculus most of the time. However, the situation changes completely if we consider that integral for all functions in a certain function space. Then the integral becomes a functional on that space, call it f. As a space let us choose $C[a,b]$. Then f is defined by,

$$f(x) = \int_a^b x(t)\, dt$$

Where ,

$x \in C[a, b]$.

f is linear. We prove that f is bounded and has norm $\|f\| = b - a$.

We are taking $J = [a, b]$ and remembering the norm on $C[a, b]$, we obtain

$$|f(x)| = \left| \int_a^b x(t)\, dt \right|$$

$$\leqq (b-a) \max_{t \in J} |x(t)|$$

$$= (b-a)\, \|x\|.$$

Taking the supremum over all x of norm 1,

we obtain,

$$\|f\| \leqq b - a.$$

For getting,

$$\|f\| \geqq b - a,$$

We are choosing

$$x = x_0 = 1, \text{ note that } \|x_0\| = 1.$$

Since,

$$|f(x)| \leqq \|f\| \, \|x\|,$$

$$\|f\| \geqq \frac{|f(x_0)|}{\|x_0\|}$$

$$= |f(x_0)|$$

$$= \int_a^b dt = b - a.$$

7.6 DUAL SPACE

7.6.1 ALGEBRAIC DUAL SPACE

The set of all linear functional defined on a vector space X can itself be made into a vector space. This space is denoted by X^* and is called the algebraic dual space of X. Its algebraic operations of vector space are defined in a natural way as follows.

The *sum* $f_1 + f_2$ of two functionals f_1 and f_2 is the functional s whose value at every $x \in X$ is

$$s(x) = (f_1 + f_2)(x) = f_1(x) + f_2(x);$$

the *product* αf of a scalar α and a functional f is the functional p whose value at $x \in X$ is

$$p(x) = (\alpha f)(x) = \alpha f(x).$$

7.6.2 SECOND ALGEBRAIC DUAL SPACE

Consider the algebraic dual $(X^*)^*$ of X^*, whose elements are the linear functionals defined on X^*. We denote $(X^*)^*$ by $X^{**}$and call it the **second algebraic dual space** of X.

We choose the notations:

Space	General element	Value at a point
X	x	—
X^*	f	$f(x)$
X^{**}	g	$g(f)$

We can obtain a $g \in X^{**}$, which is a linear functional defined on X^*, by choosing a *fixed* $x \in X$ and setting

$$g(f) = g_x(f) = f(x) \qquad (x \in X \text{ fixed, } f \in X^* \text{ variable}).$$

……………………………………………………………(5)

The subscript, x is a little reminder that we got g by the use of a certain $x \in X$.

From equation (5),

g_x is linear. This can be seen from

$$g_x(\alpha f_1 + \beta f_2) = (\alpha f_1 + \beta f_2)(x) = \alpha f_1(x) + \beta f_2(x) = \alpha g_x(f_1) + \beta g_x(f_2).$$

Hence g_x is an element of X^{**}, by the definition of X^{**}.

7.6.3 CANONOCAL MAPPING

To each $x \in X$ there corresponds a $g_x \in X^{**}$. This defines a mapping

$$C\colon X \longrightarrow X^{**}$$
$$x \longmapsto g_x.$$

C is called the **canonical mapping** of X into X^{**}.

C is linear since its domain is a vector space and we have

$$\begin{aligned}(C(\alpha x+\beta y))(f) &= g_{\alpha x+\beta y}(f)\\ &= f(\alpha x+\beta y)\\ &= \alpha f(x)+\beta f(y)\\ &= \alpha g_x(f)+\beta g_y(f)\\ &= \alpha(Cx)(f)+\beta(Cy)(f).\end{aligned}$$

C is also called the *canonical embedding* of X into X^{**}.

7.6.4 ISOMPRPHISM

By definition, this is a bijective mapping of X onto $\tilde{X}$ which preserves the structure.

Accordingly, an *isomorphism T of a metric space* $X=(X, d)$ onto a metric space $\tilde{X}=(\tilde{X}, \tilde{d})$ is a bijective mapping which preserves distance, that is, for all $x, y \in X$,

$$\tilde{d}(Tx, Ty) = d(x, y).$$

$\tilde{X}$ is then called *isomorphic* with X.

An *isomorphism T of a vector space* X onto a vector space $\tilde{X}$ over the same field is a bijective mapping which preserves the two algebraic operations of vector space; thus, for all $x, y \in X$ and scalars α,

$$T(x+y) = Tx+Ty, \qquad T(\alpha x) = \alpha Tx,$$

that is, $T: X \longrightarrow \tilde{X}$ is a bijective linear operator. $\tilde{X}$ is then called *isomorphic* with X, and X and $\tilde{X}$ are called *isomorphic vector spaces.*

If X is isomorphic with a subspace of a vector space Y, we say that X is **embeddable** in Y. Hence X is embeddable in X^{**}, and C is also called the *canonical embedding* of X into X^{**}.

If C is surjective (hence bijective), so that $\mathcal{R}(C) = X^{**}$, then X is said to be **algebraically reflexive.**

(Null space) The *null space* $N(M^*)$ of a set $M^* \subset X^*$ is defined to be the set of all $x \in X$ such that $f(x) = 0$ for all $f \in M^*$.

(Hyperplane) If Y is a subspace of a vector space X and codim $Y = 1$ then every element of X/Y is called a *hyperplane parallel to Y.*

(Half space) Let $f \neq 0$ be a bounded linear functional on a real normed space X. Then for any scalar c we have a hyperplane $H_c = \{x \in X \mid f(x) = c\}$, and H_c determines the two *half spaces*

$$X_{c1} = \{x \mid f(x) \leqq c\} \qquad \text{and} \qquad X_{c2} = \{x \mid f(x) \geqq c\}.$$

7.7 LINEAR OPERATORS AND FUNCTIONAL ON FINITE DIMENSIONAL SPACES

Matrices become the most important tools for studying linear operators in the finite dimensional case. In this connection we should also remember Theorem : If a normed space X is finite dimensional, then every linear operator on X is bounded. To understand the full significance of our present consideration.

Let X and Y be finite dimensional vector spaces over the same field and $T: X \longrightarrow Y$ a linear operator. We choose a basis $E = \{e_1, \cdots, e_n\}$ for X and a basis $B = \{b_1, \cdots, b_r\}$ for Y, with the vectors arranged in a definite order which we keep fixed. Then every $x \in X$ has a unique representation

$$x = \xi_1 e_1 + \cdots + \xi_n e_n.$$

……………………………..(6)

Since T is linear, x has the image

$$y = Tx = T\left(\sum_{k=1}^{n} \xi_k e_k\right) = \sum_{k=1}^{n} \xi_k T e_k.$$

…………………………………………………..(7)

Since the representation (6) is unique, we have our first result:

T is uniquely determined if the images $y_k = Te_k$ of the n basis vectors $e_1, \cdots, e_n$ are prescribed.

Since y and $y_k = Te_k$ are in Y, they have unique representations of the form

$$y = \sum_{j=1}^{r} \eta_j b_j$$

…………………..(8)

$$Te_k = \sum_{j=1}^{r} \tau_{jk} b_j.$$

………………………..(9)

Substitution into (7) gives,

$$y = \sum_{j=1}^{r} \eta_j b_j = \sum_{k=1}^{n} \xi_k Te_k = \sum_{k=1}^{n} \xi_k \sum_{j=1}^{r} \tau_{jk} b_j = \sum_{j=1}^{r} \left(\sum_{k=1}^{n} \tau_{jk} \xi_k \right) b_j.$$

Since the b_j's form a linearly independent set, the coefficients of each b_j on the left and on the right must be the same, that is,

$$\eta_j = \sum_{k=1}^{n} \tau_{jk} \xi_k \qquad j = 1, \cdots, r.$$

…………………………………………………………..(10)

This yields our next result:

The image $y = Tx = \sum \eta_j b_j$ of $x = \sum \xi_k e_k$ can be obtained from (10). The coefficients in (10) form a matrix.

with r rows and n columns. If a basis E for X and a basis B for Y are given, with the elements of E and B arranged in some definite order (which is arbitrary but fixed), then the matrix T_{EB} is uniquely determined by the linear operator T. We say that the matrix T_{EB} **represents** the operator T with respect to those bases.

By introducing the column vectors $\tilde{x} = (\xi_k)$ and $\tilde{y} = (\eta_j)$ we can write (10) in matrix notation:

$$\tilde{y} = T_{EB}\tilde{x}. \qquad \text{……………………….(11)}$$

Similarly (09) can also be written- in matrix notation,

$$Te = T_{EB}{}^{\mathsf{T}} b \qquad \text{………………(12)}$$

where Te is the column vector with components $Te_1, \cdots, Te_n$ (which are themselves vectors) and b is the column vector with components

$b_1, \cdots, b_r$, and we have to use the transpose $T_{EB}{}^{\mathsf{T}}$ of T_{EB}

Our consideration shows that a linear operator T determines a uniqe matrix representing T with respect to a given basis for X and a given basis for Y, where the vectors of each of the bases are assumed to be arranged in a fixed order. Conversely, any matrix with r rows and n columns determines a linear operator which it represents with respect to given bases for X and Y.

Let us now turn to **linear functionals** on X, where $\dim X = n$ and $\{e_1, \cdots, e_n\}$ is a basis for X, as before. These functionals constitute the algebraic dual space X^* of X, as we know from the previous section For every such functional f and every $x = \sum \xi_j e_j \in X$ we have

$$f(x) = f\left(\sum_{j=1}^{n} \xi_j e_j\right) = \sum_{j=1}^{n} \xi_j f(e_j) = \sum_{j=1}^{n} \xi_j \alpha_j$$

………………………………………………..(13)

Where,

$$\alpha_j = f(e_j) \qquad\qquad j = 1, \cdots, n,$$

………………………………………….(14)

and f is uniquely determined by its values α_j at the n basis vectors of X.

Conversely, every n-tuple of scalars $\alpha_1, \cdots, \alpha_n$ determines a linear functional on X. In particular, let us take the n-tuples ,

$$
\begin{array}{cccccc}
(1, & 0, & 0, & \cdots & 0, & 0) \\
(0, & 1, & 0, & \cdots & 0, & 0) \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
(0, & 0, & 0, & \cdots & 0, & 1).
\end{array}
$$

By (13) and (14),

this gives n functionals, which we denote by $f_1, \cdots, f_n$, with

$$f_k(e_j) = \delta_{jk} = \begin{cases} 0 & \text{if } j \neq k, \\ 1 & \text{if } j = k; \end{cases}$$

…………………………………………..(15).

that is, f_k has the value 1 at the kth basis vector and 0 at the $n-1$ other basis vectors. δ_{jk} is called the *Kronecker delta.* $\{f_1, \cdots, f_n\}$ is called the **dual basis** of the basis $\{e_1, \cdots, e_n\}$ for X.

7.7.1 THEOREMS

Theorem 1:

Theorem (Dimension of X^*). *Let X be an n-dimensional vector space and $E = \{e_1, \cdots, e_n\}$ a basis for X. Then $F = \{f_1, \cdots, f_n\}$ given by*

Equation (15) is a basis for the algebraic dual X^* of X, and $dimX^* = dimX = n$.

Proof.

F is a linearly independent set since

$$\sum_{k=1}^{n} \beta_k f_k(x) = 0 \qquad (x \in X)$$

……………………………………………………..(16).

with $x = e_j$ gives

$$\sum_{k=1}^{n} \beta_k f_k(e_j) = \sum_{k=1}^{n} \beta_k \delta_{jk} = \beta_j = 0,$$

So that all the β_k's (16) are zero. We show that every $f \in X^*$ can be represented as a linear combination of the elements of F in a unique way.

We write $f(e_j) = \alpha_j$

By using (13) and (14),

$$f(x) = \sum_{j=1}^{n} \xi_j \alpha_j$$

……………………..(17)

for every $x \in X$.

.

$$f_j(x) = f_j(\xi_1 e_1 + \cdots + \xi_n e_n) = \xi_j.$$

Together,

$$f(x) = \sum_{j=1}^{n} \alpha_j f_j(x).$$

Hence the unique representation of the arbitrary linear functional f on X in terms of the functionals $f_1, \cdots, f_n$ is

$$f = \alpha_1 f_1 + \cdots + \alpha_n f_n.$$

Lemma 1:

Let X be a finite dimensional vector space. If $x_0 \in X$ has the property that $f(x_0) = 0$ for all $f \in X^*$, then $x_0 = 0$.

Proof. Let $\{e_1, \cdots, e_n\}$ be a basis for X and $x_0 = \sum \xi_{0j} e_j$.

From (13) and (14),

$$f(x_0) = \sum_{j=1}^{n} \xi_{0j} \alpha_j.$$

By assumption this is zero for every $f \in X^*$, that is, for every choice of $\alpha_1, \cdots, \alpha_n$. Hence all ξ_{0j} must be zero.

Theorem 2:

Theorem (Algebraic reflexivity). *A finite dimensional vector space is algebraically reflexive.*

Proof. The canonical mapping $C: X \longrightarrow X^{**}$ considered in the previous section is linear. $Cx_0 = 0$ means that for all $f \in X^*$ we have

$$(Cx_0)(f) = g_{x_0}(f) = f(x_0) = 0,$$

by the definition of C. This implies $x_0 = 0$ by Lemma previous , Hence from **Theorem 2** of **unit -6** it follows that the mapping C has an inverse ,

$C^{-1}: \mathcal{R}(C) \longrightarrow X$, where $\mathcal{R}(C)$ is the range of C. We also have $\dim \mathcal{R}(C) = \dim X$ by the same theorem.

From previous theorem,

$$\dim X^{**} = \dim X^* = \dim X.$$

Together, $\dim \mathcal{R}(C) = \dim X^{**}$. Hence $\mathcal{R}(C) = X^{**}$ because $\mathcal{R}(C)$ is a vector space and a proper subspace of X^{**} has dimension less than dim X^{**},[We are using the theorem: let X be an n dimensional vector space. Then any proper subspace Y of X has dimension less than n]. By the definition, this proves algebraic reflexivity.

7.8 NORMED SPACES OF OPERATORS. DUAL SPACE

We take any two normed spaces X and Y (both real or both complex) and consider the set $B(X, Y)$, consisting of all bounded linear operators from X into Y, that is, each such operator is defined on all of X and its range lies in Y.

Theorem 3:

The vector space $\mathbf{B}(X, \mathbf{Y})$ *of all bounded linear operators from a normed space* $\mathbf{X}$ *into a normed space* $\mathbf{Y}$ *is itself a normed space with norm defined by*

$$\|T\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x \in X \\ \|x\|=1}} \|Tx\|.$$

…………………………………………………..(18)

Theorem 4:

If $\mathbf{Y}$ *is a Banach space, then* $\mathbf{B}(X, Y)$ *is a Banach space.*

Proof. We consider an arbitrary Cauchy sequence (T_n) in $B(X, Y)$ and show that (T_n) converges to an operator $T \in B(X, Y)$.

Since (T_n) is Cauchy, for every $\varepsilon > 0$ there is an N such that

$$\|T_n - T_m\| < \varepsilon \qquad (m, n > N).$$

For all $x \in X$ and $m, n > N$ we thus obtain

$$\|T_n x - T_m x\| = \|(T_n - T_m)x\| \leqq \|T_n - T_m\| \, \|x\| < \varepsilon \|x\|.$$

…………………………………………………………………(19)

Now for any fixed x and given $\tilde{\varepsilon}$ we may choose $\varepsilon = \varepsilon_x$ so that

$\varepsilon_x \, \|x\| < \tilde{\varepsilon}$.

Then from (19),

we have $\|T_n x - T_m x\| < \tilde{\varepsilon}$ and see that $(T_n x)$ is

Cauchy in Y. Since Y is complete, $(T_n x)$ converges, say, $T_n x \longrightarrow y$. Clearly, the limit $y \in Y$ depends on the choice of $x \in X$. This defines an operator $T\colon X \longrightarrow Y$, where $y = Tx$. The operator T is linear since

$$\lim T_n(\alpha x + \beta z) = \lim (\alpha T_n x + \beta T_n z) = \alpha \lim T_n x + \beta \lim T_n z.$$

We prove that T is bounded and $T_n \longrightarrow T$, that is, $\|T_n - T\| \longrightarrow 0$.

for every $m > N$ and $T_m x \longrightarrow Tx$, we may let

[Using the equation (19)]

$m \longrightarrow \infty$. Using the continuity of the norm, we then obtain from

equation (19),

for every $n > N$ and all $x \in X$

$$\|T_n x - Tx\| = \|T_n x - \lim_{m\to\infty} T_m x\| = \lim_{m\to\infty} \|T_n x - T_m x\| \leqq \varepsilon \|x\|.$$

…………………………………………………………………………(20)

This shows that $(T_n - T)$ with $n > N$ is a bounded linear operator. Since T_n is bounded, $T = T_n - (T_n - T)$ is bounded, that is,

$T \in B(X, Y)$.

Furthermore, if in (20) we take the supremum over all x of norm 1, we obtain,

$$\|T_n - T\| \leqq \varepsilon \qquad (n > N).$$

Hence $\|T_n - T\| \longrightarrow 0.$

Definition (Dual space X'). Let X be a normed space. Then the set of all bounded linear functionals on X constitutes a normed space with norm defined by

$$\|f\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{|f(x)|}{\|x\|} = \sup_{\substack{x \in X \\ \|x\|=1}} |f(x)|$$

which is called the dual space of X and is denoted by X′.

Theorem 4.

The dual space X' of a normed space X is a Banach space (*whether or not X is*).

An **isomorphism** of a normed space X onto a normed space $\tilde{X}$ is a bijective linear operator T: $X \longrightarrow \tilde{X}$ which preserves the norm, that is, for all $x \in X$,

$$\|Tx\| = \|x\|.$$

(Hence T is isometric.) X is then called *isomorphic* with $\tilde{X}$, and X and $\tilde{X}$ are called *isomorphic normed spaces.*—From an abstract point of view, X and $\tilde{X}$ are then identical, the isomorphism merely amounting to renaming of the elements (attaching a "tag" T to each point).

- If a normed space X is finite dimensional, then every linear operator on X is bounded. This result using also equation (13) and (14) are using. ………………………(a)

Examples:

1. Space $\mathbb{R}^n$: The dual space of $\mathbb{R}^n$ **is** $\mathbb{R}^n$.

Proof. We have $\mathbf{R}^{n\prime} = \mathbf{R}^{n*}$ by

Using above result (a),

$$f(x) = \sum \xi_k \gamma_k \qquad\qquad \gamma_k = f(e_k)$$

(sum from 1 to n). By the Cauchy-Schwarz inequality

$$|f(x)| \leqq \sum |\xi_k \gamma_k| \leqq \left(\sum \xi_j^2\right)^{1/2} \left(\sum \gamma_k^2\right)^{1/2} = \|x\| \left(\sum \gamma_k^2\right)^{1/2}.$$

Taking the supremum over all x of norm 1 we obtain

$$\|f\| \leqq \left(\sum \gamma_k^2\right)^{1/2}.$$

However, since for $x = (\gamma_1, \cdots, \gamma_n)$ equality is achieved in the Cauchy-Schwarz inequality, we must in fact have

$$\|f\| = \left(\sum_{k=1}^{n} \gamma_k^2\right)^{1/2}.$$

This proves that the norm of f is the Euclidean norm, and $\|f\| = \|c\|$, where $c = (\gamma_k) \in \mathbf{R}^n$. Hence the mapping of $\mathbf{R}^{n\prime}$ onto $\mathbf{R}^n$ defined by $f \longmapsto c = (\gamma_k)$, $\gamma_k = f(e_k)$, is norm preserving and, since it is linear and bijective, it is an isomorphism.

2. Space l^1: The dual space of l^1 is l^∞.

Proof. A Schauder basis for l^1 is (e_k), where $e_k = (\delta_{kj})$ has 1 in the kth place and zeros otherwise. Then every $x \in l^1$ has a unique representation

$$x = \sum_{k=1}^{\infty} \xi_k e_k.$$

…………………………………………………….(21)

We consider any $f \in l^{1\prime}$, where $l^{1\prime}$ is the dual space of l^1. Since f is linear and bounded,

$$f(x) = \sum_{k=1}^{\infty} \xi_k \gamma_k \qquad \gamma_k = f(e_k)$$

....(22)

where the numbers $\gamma_k = f(e_k)$ are uniquely determined by f. Also $\|e_k\| = 1$ and

$$|\gamma_k| = |f(e_k)| \leqq \|f\| \, \|e_k\| = \|f\|, \qquad \sup_k |\gamma_k| \leqq \|f\|.$$

....(23)

Hence $(\gamma_k) \in l^\infty$.

On the other hand, for every $b = (\beta_k) \in l^\infty$ we can obtain a corresponding bounded linear functional g on l^1. In fact, we may define g on l^1 by

$$g(x) = \sum_{k=1}^{\infty} \xi_k \beta_k$$

where $x = (\xi_k) \in l^1$. Then g is linear, and boundedness follows from

$$|g(x)| \leqq \sum |\xi_k \beta_k| \leqq \sup_j |\beta_j| \sum |\xi_k| = \|x\| \sup_j |\beta_j|$$

(sum from 1 to ∞). Hence $g \in l^{1\prime}$.

We finally show that the norm of f is the norm on the space l^∞.

From (22) we have

$$|f(x)| = \left|\sum \xi_k \gamma_k\right| \leqq \sup_j |\gamma_j| \sum |\xi_k| = \|x\| \sup_j |\gamma_j|.$$

Taking the supremum over all x of norm 1, we see that

$$\|f\| \leqq \sup_j |\gamma_j|.$$

From this and (23),

$$\|f\| = \sup_j |\gamma_j|,$$
.....................(24)

which is the norm on l^∞. Hence this formula can be written $\|f\| = \|c\|_\infty$, where $c = (\gamma_j) \in l^\infty$. It shows that the bijective linear mapping of $l^{1\prime}$ onto l^∞ defined by $f \longmapsto c = (\gamma_j)$ is an isomorphism.

3.

Space l^p. *The dual space of l^p is l^q; here, $1 < p < +\infty$ conjugate of p, that is, $1/p + 1/q = 1$.*

Proof. A Schauder basis for l^p is (e_k), where $e_k = (\delta_{kj})$ as in the preceding example. Then every $x \in l^p$ has a unique representation

$$x = \sum_{k=1}^{\infty} \xi_k e_k.$$

....(25)

We consider any $f \in l^{p\prime}$, where $l^{p\prime}$ is the dual space of l^p. Since f is linear and bounded,

$$f(x) = \sum_{k=1}^{\infty} \xi_k \gamma_k \qquad \gamma_k = f(e_k).$$

....(26)

Let q be the conjugate of p and consider $x_n = (\xi_k^{(n)})$ with

$$\xi_k^{(n)} = \begin{cases} |\gamma_k|^q/\gamma_k & \text{if } k \leqq n \text{ and } \gamma_k \neq 0, \\ 0 & \text{if } k > n \text{ or } \gamma_k = 0. \end{cases}$$
....(27)

By substituting this into (26) we obtain

$$f(x_n) = \sum_{k=1}^{\infty} \xi_k^{(n)} \gamma_k = \sum_{k=1}^{n} |\gamma_k|^q.$$

We also have, using (27) and $(q-1)p = q$,

$$f(x_n) \leqq \|f\| \, \|x_n\| = \|f\| \left(\sum |\xi_k^{(n)}|^p \right)^{1/p}$$

$$= \|f\| \left(\sum |\gamma_k|^{(q-1)p} \right)^{1/p}$$

$$= \|f\| \left(\sum |\gamma_k|^q \right)^{1/p}$$

(sum from 1 to n). Together,

$$f(x_n) = \sum |\gamma_k|^q \leqq \|f\| \left(\sum |\gamma_k|^q \right)^{1/p}.$$

Dividing by the last factor and using $1 - 1/p = 1/q$, we get

$$\left(\sum_{k=1}^{n} |\gamma_k|^q \right)^{1-1/p} = \left(\sum_{k=1}^{n} |\gamma_k|^q \right)^{1/q} \leqq \|f\|.$$

Since n is arbitrary, letting $n \longrightarrow \infty$, we obtain

$$\left(\sum_{k=1}^{\infty} |\gamma_k|^q \right)^{1/q} \leqq \|f\|.$$

....(28).

Hence $(\gamma_k) \in l^q$.

Conversely, for any $b = (\beta_k) \in l^q$ we can get a corresponding bounded linear functional g on l^p. In fact, we may define g on l^p by setting

$$g(x) = \sum_{k=1}^{\infty} \xi_k \beta_k$$

where $x = (\xi_k) \in l^p$. Then g is linear, and boundedness follows from the Hölder inequality Hence $g \in l^{p\prime}$.

We finally prove that the norm of f is the norm on the space l^q.

$$|f(x)| = |\sum \xi_k \gamma_k| \leqq \left(\sum |\xi_k|^p\right)^{1/p} \left(\sum |\gamma_k|^q\right)^{1/q}$$

$$= \|x\| \left(\sum |\gamma_k|^q\right)^{1/q}$$

Using Holder inequality and from equation (26),

(sum from 1 to ∞); hence by taking the supremum over all x of norm 1 we obtain

$$\|f\| \leqq \left(\sum |\gamma_k|^q\right)^{1/q}.$$

From (27) we see that the equality sign must hold, that is,

$$\|f\| = \left(\sum_{k=1}^{\infty} |\gamma_k|^q\right)^{1/q}. \qquad \text{.............................(29)}$$

This can be written $\|f\| = \|c\|_q$, where $c = (\gamma_k) \in l^q$ and $\gamma_k = f(e_k)$. The mapping of $l^{p\prime}$ onto l^q defined by $f \longmapsto c$ is linear and bijective, and from (29) we see that it is norm preserving, so that it is an isomorphism.

7.9 SUMMARY

We explained in this unit the concept of Linear Functional, Linear Operators, Functional on Finite Dimensional Spaces, Normed Spaces of Operators and Dual Space.

6.11 GLOSSARY

i. **Metric space:** Let $X \neq \emptyset$ be a set then the metric on the set X is defined as a function $d: X \times X \to [0, \infty)$ such that some conditions are satisfied.

ii. **Vector space: -** Let V be a nonempty set with two operations

(i) **Vector addition:** If any $u, v \in V$ then $u + v \in V$

(ii) **Scalar Multiplication:** If any $u \in V$ and $k \in F$ then $ku \in V$

Then V is called a vector space (over the field F) if the following axioms hold for any vectors if the some conditions hold.

iii. **Normed space:-** Let X be a vector space over scalar field K. A *norm* on a (real or complex) vector space X is a real-valued function on X ($\|x\|: X \to K$) whose value at an $x \in X$ is denoted by $\|x\|$ and which has the four properties here x and y are arbitrary vectors in X and α is any scalar.

iv. **Banach space:-** A complete normed linear space is called a Banach space.

v. **Linear operator:-** A *linear operator* T is an operator such that

i. The domain $\mathcal{D}(T)$ of T is a vector space and the range $\mathcal{R}(T)$ lies in a vector space over the same field.

ii. for all $x, y \in \mathcal{D}(T)$ and scalar α,

$$T(x + y) = Tx + Ty,$$

$$T(\alpha x) = \alpha Tx.$$

vi. **Bounded linear operator:-** Let X and Y be normed spaces. Let $T: \mathcal{D}(T) \to Y$ be a linear operator with domain $\mathcal{D}(T) \subset X$.

The operator T is said to be bounded if there is a real number c such that for all $x \in \mathcal{D}(T)$,

$\|Tx\| \leq c\|x\|$.

vii. Quotient Spaces

viii. Subspace

ix. Finite dimensional Normed Spaces

x. Equivalent norms

CHECK YOUR PROGRESS

1. *The dual space of* ℓ^1 *is* ....

2. *The dual space of* ℓ^p *is* ....

here $1 < p < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1.$

3. *The dual space of* c_0 *is* ....

4. If f is a linear functional on a normed space X, then $\ker f$ is:

(a) closed in X.

(b) dense in X.

(c) either closed or dense in X.

(d) None of these.

5. Let X be a normed space and f be a bounded, non-zero linear functional on X. Then, which of the following is not true?

(a) f is onto.

(b) f is continuous.

(c) $\ker f$ is a closed subspace of X.

(d) f is an open map.

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7.11 SUGGESTED READINGS

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iii. B. V. Limaye, (2014), *Functional Analysis*, New age International Private Limited.

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6.13 TERMINAL QUESTIONS

Q.1.

Let X be a normed space and f, g are nonzero linear functionals on X. Show that

$$\ker(f) = \ker(g) \iff f = cg \ \ \textit{for some nonzero scalar} \ \ c.$$

Q.2.

Let X be a normed space and f are nonzero linear functional on X. Show that f is continuous if and only if $\ker(f)$ is closed.

Q.3 Define linear functional……………………………………….

Q.4 Define the difference between linear operator and linear functional…..

6.14 ANSWERS

CHECK YOUR PROGRESS

1. ℓ^∞.
2. ℓ^q.
3. ℓ^1.
4. (c)
5. (d)

BLOCK III: INNER PRODUCT SPACE AND HILBERT SPACE

UNIT 8:

INNER PRODUCT SPACE AND HILBERT SPACE

CONTENTS:

8.1 INTRODUCTION

In the preceding units, we discussed normed linear spaces and Banach spaces. These spaces have linear properties as well as metric properties. Although the norm on a linear space generalizes the elementary concept of the length of a vector, but the main geometric concept other than the length of a vector is the angle between two vectors, In this unit, we take the opportunity to study linear spaces having an inner product, a generalization of the usual dot product on finite dimensional linear spaces. The concept of an inner product in a linear space leads to an inner product space and a complete inner product space which is called a Hilbert space. The theory of Hilbert Spaces does not deal with angles in general. Most interestingly, it helps us to introduce an idea of perpendicularity for two vectors and the geometry deals in various fundamental aspects with Euclidean geometry.

The basic of the theory of Hilbert spaces was given by in 1912 by the work of German mathematician D. Hilbert (1862 -1943) on integral equations. However, an axiomatic basis of the theory was given by famous mathematician J. Von Neumann (1903 -1957). However, Hilbert spaces are the simplest type of infinite dimensional Banach spaces to tackle a remarkable role in functional analysis.

8.2 OBJECTIVES

After studying this unit, learner will be able to

- **i.** Described the concept of *InnerProduct space.*
- **ii.** Described the concept of *Hilbert space.*
- **iii.** Problems and examples related to *InnerProduct space* and *Hilbert space.*

8.3 BASICS

We first defined the basic orientations:

8.3.1 INNER PRODUCT SPACE AND HILBERT SPACE

An inner product space X is a vector space with an inner product $\langle x, y \rangle$ defined on it. The latter generalizes the dot product of vectors in three dimensional space and is used to define

I. A norm $\|.\|$by$\|x\| = \langle x, x \rangle^{1/2}$,

II. Orthogonality by $\langle x, y \rangle = 0$.

A Hilbert space ***H*** is a complete inner product space. The theory of inner product and Hilbert spaces is richer than that of general normed and Banach spaces. Distinguishing features are

i. Representations of ***H*** as a direct sum of a closed subspace and its orthogonal complement.

ii. Orthonormal sets and sequences and corresponding representations of elements of ***H***.

iii. The Riesz representation of bounded linear functional by inner products.

iv. The Hilbert-adjoint operator *T** of a bounded linear operator *T*.

Orthonormal sets and sequences are truly interesting only if they are total. Hilbert-adjoint operators can be used to define classes of operators which are of great importance in applications.

8.3.2 INNER PRODUCT SPACE, HILBERT SPACE

Definition (Inner product space, Hilbert space) -An inner product space (or pre-Hilbert space) is a vector space X with an inner product defined on X. A Hilbert space is a complete inner product space (complete in the metric defined by the inner product). Here, an inner product on X is a mapping of $X \times X$ into the scalar field K of X; that is, with every pair of vectors x and y there is associated a scalar which is written$\langle x, y\rangle$ and is called the inner product of x and y, such that for all vectors x, y, z and scalars α we have

(lPl)$\langle x+y,\ z\rangle = \langle x,\ z\rangle + \langle y, z\rangle$

(lP2)$\langle \alpha x,\ y\rangle = \ \alpha\langle x,\ y\rangle$

(IP3)$\langle x, y\rangle = \ \overline{\langle y, x\rangle}, \langle x, x\rangle \geq 0$

(IP4)$\langle x, x\rangle = 0 \ \ \Leftrightarrow \ \ x = 0.$

An inner product on X defines a norm on X given by

(1)$\|x\| = \langle x, x\rangle^{1/2} \qquad (\geq 0)$

and a metric on$X$ given by

(2)$d(x, y) = \|x-y\| = \ \sqrt{\langle x-y,\ x-y\rangle}$

Hence inner product spaces are normed spaces, and Hilbert spaces are Banach spaces.

In (IP3), the bar denotes complex conjugation. Consequently, if X is a real vector space, we simply have

$\langle x, y\rangle = \ \langle y, x\rangle$ (Symmetry).

From (IP1) to (IP3) we obtain the formula

(3) (a) $\langle \alpha x + \beta y,\ z\rangle = \alpha\langle x, z\rangle + \beta\langle y,\ z\rangle$

(b) $\langle x, \alpha y\rangle = \bar{\alpha}\,\langle x, y\rangle$

(c)$\langle x,\ \alpha y + \beta z\rangle = \bar{\alpha}\,\langle x, y\rangle + \bar{\beta}\langle x,\ z\rangle$

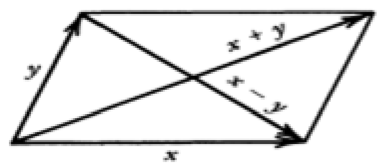
which we shall use quite often. (3a) shows that the inner product is linear in the first factor. Since in (3c) we have complex conjugates $\bar{\alpha}$ and $\bar{\beta}$ on the right, we say that the inner product is conjugate linear in the second factor. Expressing both properties together, we say that the inner product is sesquilinear. This means "$1\frac{1}{2}$ times linear" and is motivated by the fact that "conjugate linear" is also known as "semilinear" (meaning "halftinear"), a less suggestive term which we shall not use. The reader may show by a simple straightforward calculation that a norm on an inner product space satisfies the important **parallelogram equality**

(4)$\|\boldsymbol{x} + \boldsymbol{y}\|^2 + \|\boldsymbol{x} - \boldsymbol{y}\|^2 = \boldsymbol{2}(\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2).$

This name is suggested by elementary geometry, as we see from Fig. 23 if we remember that the norm generalizes the elementary concept of the length of a vector . It is quite remarkable that such an equation continues to hold in our present much more general setting. We conclude that if a norm does not satisfy (4), it cannot be obtained from an inner product by the use of (1). Such norms do exist; examples will be given below. Without risking misunderstandings we may thus say:

Not all normed spaces are inner product spaces.

Before we consider examples, let us define the concept of orthogonality, which is basic in the whole theory. We know that if the dot product of two vectors in three dimensional spaces is· zero, the vectors are orthogonal, that is, they are perpendicular or at least one of them is the zero vector. This suggests and motivates the following



Parallelogram with sides x and y in the plane

Fig.8.3.2

8.3.3 ORTHOGONALITY

An element x of an inner product space X is said to be orthogonal to an element $y \in X$if

$$\langle x, y \rangle = 0.$$

We also say that x and y are orthogonal, and we write $x \perp y$. Similarly, for subsets $A, B \subset X$we write$x \perp A$ if $x \perp a$for all $a \in A$, and $A \perp B$ if $a \perp b$for all $a \in A$ and all $b \in B$.

8.4.0 RESULTSANDEXAMPLES

8.4.1 RESULTS

Theorem 8.4.1: Every inner product function is a continuous function. (Equivalently, if $f : X \times X \to \mathbb{C}$ defined by $f(x, y) = \langle x, y\rangle, \forall\, x, y \in X$ *then* f *is continuous*).

Proof: Let X be an inner product space. Define $f : X \times X \to \mathbb{C}$ by $f(x, y) = \langle x, y\rangle, \forall\, x, y \in X$. Now take $\{x_n\}$ and $\{y_n\}$ be a sequence in X such that $x_n \to x$ as $n \to \infty$ and $y_n \to y$ as $n \to \infty$.

So, $||x_n - x|| \to 0$ as $n \to \infty$ and $||y_n - y|| \to 0$ as $n \to \infty$. As, $x_n \to x$ as $n \to \infty$ then, $||x_n|| \to ||x||$ as $n \to \infty$. So, $\{||x_n||\}$ are bounded. So, there exists a constant $M > 0$ such that $||x_n|| \le M, \forall\, n$.

Now, $\left|\langle x_n, y_n\rangle - \langle x, y\rangle\right|$

$$= \left|\langle x_n, y_n\rangle - \langle x_n, y\rangle + \langle x_n, y\rangle - \langle x, y\rangle\right|$$

$$= |\langle x_n, y_n - y\rangle + \langle x_n - x, y\rangle| \le |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle|$$

$\le ||x_n||\,||y_n - y|| + ||x_n - x||\,||y||$ [By C-S inequality]

$$\le M\left||y_n - y|\right| + \left||x_n - x|\right|\left||y|\right|$$

$\to 0$ as $n \to \infty$

i.e $\langle x_n, y_n\rangle \to \langle x, y\rangle$ *as* $n \to \infty$, implying that $f(x_n, y_n) \to f(x, y)$ as $n \to \infty$. So, f is continuous.

Theorem 8.4.2 (Parallelogram Law): Let X be an inner product space and let x, y ∈ X. Then,

$$\left||x + y|\right|^2 + \left||x - y|\right|^2 = 2(\left||x|\right|^2 + \left||y|\right|^2)$$

Proof: $\left||x + y|\right|^2 = \langle x + y, x + y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle$

$= \left||x|\right|^2 + ||y|^2| + \langle x, y\rangle + \langle y, x\rangle$ **(8.4.3)** *and* $\left||x - y|\right|^2 = \langle x - y, x - y\rangle$

$$= \langle x, x\rangle + \langle x, -y\rangle + \langle -y, x\rangle + \langle -y, -y\rangle$$

$= \left||x|\right|^2 + \left||y|\right|^2 - \langle x, y\rangle - \langle y, x\rangle$ **(8.4.4)**

Adding (8.4.3) and (8.4.4) we get

$$||x + y||^2 + \|x - y\|^2 = 2(||x||^2 + ||y||^2)$$

Theorem 8.4.5(Polarization Identity): Let X be an inner product space, let $x, y \in X$ then,

$$\langle x, y \rangle = \frac{1}{4}[\|x + y\|^2 - \|x - y\|^2 + i\,\|x + iy\|^2 - \|x - iy\|^2]$$

Proof: Now, $||x + y||^2 = \left||x|\right|^2 + \left||y|\right|^2 + \langle x, y \rangle + \langle y, x \rangle$ **(8.4.6)**

$\|x - y\|^2 = ||x||^2 + ||y||^2 - \langle x, y \rangle - \langle y, x \rangle$**(8.4.7)**

Replacing y by iy in (8.4.6) and (8.4.7)

$\|x + iy\|^2 = ||x||^2 + ||iy||^2 + \langle x, iy \rangle + \langle iy, x \rangle$

$= ||x||^2 + ||y||^2 - i\langle x, y \rangle + i\langle y, x \rangle$**(8.4.8)**

$\|x - iy\|^2 = ||x||^2 + ||iy||^2 - \langle x, iy \rangle - \langle iy, x \rangle$

$= ||x||^2 + ||y||^2 + i\langle x, y \rangle - i\langle y, x \rangle$**(8.4.9)**

(8.4.6) – (8.4.7) + i(8.4.8) –i(8.4.9), we get the required result.

8.4.2 EXAMPLES

Example 1: The Euclidean space $\mathbb{R}^n$ is a Hilbert space.

Solutions:-Euclidean space $\mathbb{R}^n$- The space $\mathbb{R}^n$ is a Hilbert space with inner product defined by

(5) $\langle x, y \rangle = \xi_1\eta_1 + \cdots \ldots \xi_n\eta_n$

Where $x = (\xi_j) = (\xi_1, \ldots \ldots . \xi_n)$ and $y = \eta_j = (\eta_1, \ldots \ldots . . \eta_n)$.

In fact, from (5) we obtain

$$\|x\| = \langle x, x \rangle^{1/2} = \left(\xi_1{}^2 + \cdots \ldots \ldots + \xi_n{}^2\right)^{1/2}$$

And from this the Euclidean metric defined by

$$d(x, y) = \|x - y\| = \langle x - y, x - y\rangle^{1/2}$$

$$= [(\xi_1 - \eta_1)^2 + \cdots \ldots (\xi_n - \eta_n)^2]^{1/2}$$

If n = 3, formula (5) gives the usual dot product

$$\langle x, y\rangle = x \cdot y = \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3$$

Of $x = (\xi_1\xi_2\xi_3)$and $y = (\eta_1\eta_2\eta_3)$, and the orthogonality

$$\langle x, y\rangle = x \cdot y = 0$$

agrees with the elementary concept of perpendicularity.

Example 2:- The Euclidean space $\mathbb{C}^n$ is a Hilbert space.

Solution: -Unitary space $\mathbb{C}^n$ - The space $\mathbb{C}^n$ is a Hilbert space with inner product given by

(6) $$\langle x, y\rangle = \xi_1\overline{\eta_1} + \ldots\ldots\xi_n\overline{\eta_n}$$

In fact, from (6) we obtain the norm defined by

$$\|x\| = \left(\xi_1\overline{\xi_1} + \cdots\ldots\xi_n\overline{\xi_n}\right)^{1/2} = (|\xi_1|^2 + \cdots\ldots|\xi_n|^2)^{1/2}$$

Here we also see why we have to take complex conjugates $\overline{\eta_j}$in (6); this entails$\langle y, x\rangle = \overline{\langle x, y\rangle}$, which is (IP3) , so that $\langle x, x\rangle$is real.

Example 3: The space $L^2[a, b]$ is a Hilbert space

Solutions:-Space L^2[a, b]. The norm is defined by

$$\|x\| = \left(\int_a^b x(t)^2\, dt\right)^{1/2}$$

and can be obtained from the inner product defined by

$$\langle x, y\rangle = \int_a^b x(t)y(t)\, dt.$$

(7)

In connection with certain applications it is advantageous to remove that restriction and consider complex-valued functions (keeping $t \in [a, b]$

real, as before). These functions form a complex vector space, which becomes an inner product space if we define

$$\langle x, y\rangle = \int_a^b x(t)\overline{y(t)}\, dt. \qquad (7^*)$$

Here the bar denotes the complex conjugate. It has the effect that (IP3) holds, so that $\langle x, x\rangle$ is still real. This property is again needed in connection with the norm, which is now defined by

$$\|x\| = \left(\int_a^b |x(t)|^2\, dt\right)^{1/2}$$

Because $x(t)\overline{x(t)} = |x(t)|^2$.

The completion of the metric space corresponding to (7) is the real space $L^2[a,b]$. Similarly, the completion of the metric space corresponding to (7*) is called the complex space $L^2[a,b]$. We shall see in the next section that the inner product can be extended from an inner product space to its completion. Together with our present discussion this implies that $L^2[a,b]$ is a Hilbert space.

Example 4:Hilbert Sequence Space l^2. The space l^2 is a Hilbert space with inner product defined by

$$\langle x, y\rangle = \sum_{j=1}^{\infty} \xi_j \bar{\eta}_j. \qquad (8)$$

Convergence of this series follows from the Cauchy-Schwarz inequality and the fact that $x, y \in l^2$, by assumption. We see that (8) generalizes (6). The norm is defined by

$$\|x\| = \langle x, x\rangle^{1/2} = \left(\sum_{j=1}^{\infty} |\xi_j|^2\right)^{1/2}.$$

Completeness of l^p - The space l^p is complete.(with p=2)

Proof- Let (x_n) be any Cauchy sequence in the space l^p, where

$x_m = \left(\xi_1{}^{(m)}, \xi_2{}^{(m)}, \ldots.\right)$. Then for every $\epsilon \geq 0$ there is an N such that for all $m, n > N$,

$$d(x_m, x_n) = \left(\sum_{j=1}^{\infty} |\xi_j^{(m)} - \xi_j^{(n)}|^p\right)^{1/p} < \varepsilon.$$

(9)

It follows that for every $j = 1, 2, \ldots$we have

(10)

$$|\xi_j^{(m)} - \xi_j^{(n)}| < \varepsilon \qquad (m, n > N).$$

We choose a $fixed\ j$. From (10) we see that $\left(\xi_1{}^{(1)}, \xi_2{}^{(2)}, \ldots.\right)$ is a Cauchy sequence of numbers. It converges since $\mathbb{R}$ and $\mathbb{C}$ are complete, say, $\xi_1{}^{(m)} \to \xi_1$as $m \to \infty$. Using these limits, we define $x = (\xi_1, \xi_2, \ldots.)$ and show that and $x \in l^p$ and $x_m \to x$.

From (9) we have for all $m, n > N$

$$\sum_{j=1}^{k} |\xi_j^{(m)} - \xi_j^{(n)}|^p < \varepsilon^p \qquad (k = 1, 2, \cdots).$$

Letting $n \longrightarrow \infty$, we obtain for $m > N$

$$\sum_{j=1}^{k} |\xi_j^{(m)} - \xi_j|^p \leqq \varepsilon^p \qquad (k = 1, 2, \cdots).$$

We may now let $k \longrightarrow \infty$; then for $m > N$

$$\sum_{j=1}^{\infty} |\xi_j^{(m)} - \xi_j|^p \leqq \varepsilon^p.$$

(11)

This shows that $x_m - x = \left(\xi_j{}^{(m)} \to \xi_j\right) \in l^p$. Since $x_m \in l^p$ it follows by means of the Minkowski inequality, that

$$x = x_m + (x - x_m) \in l^p.$$

Furthermore, the series in (11) represent$[d(x_m, x)]^p$, so that (11) implies that $x_m \to x$. Since (x_m) was an arbitrary Cauchy sequence in l^p , this proves completeness of l^p, where $p = 2$ and also $1 \leq p < +\infty$.

$l^2$is the prototype of a Hilbert space. It was introduced and investigated by D. Hilbert (1912) in his work on integral equations. An axiomatic definition of Hilbert space was not given until much later, by J. von Neumann (1927), in a paper on the mathematical foundation of quantum mechanics. Cf. also J. von Neumann (1929- 30), and M. H. Stone (1932). That definition included separability, a condition which was later dropped from the definition when H. LOwig (1934), F. Rellich (1934) and F. Riesz (1934) showed that for most parts of the theory that condition was an unnecessary restriction.

Example 5: For $1 \leq p < \infty, l^p$ $(p \neq 2)$ is not an inner product space and hence not a Hilbert space

Solutions:-Our statement means that the norm l^p of with cannot be obtained from an inner product. We prove this by showing that the norm does not satisfy the parallelogram equality (4). In fact, let us take
$x = (1,1,0,0,\ldots) \in l^p$lP and $y = (1,-1,0,0,\ldots) \in l^p$ and calculate

$$\|x\| = \|y\| = 2^{1/p}, \quad \|x+y\| = \|x-y\| = 2$$

We now see that (4) is not satisfied if $\neq 2$.
$l^p$is complete. Hence l^p with$p \neq 2$ is a Banach space which is not a Hilbert space. The same holds for the space in the next example.

Example 6: The space C[a, b] is not an inner product space, hence not a Hilbert space.

Solution:-. We show that the norm defined by.

$$\|x\| = \max_{t \in J} |x(t)| \qquad J = [a, b]$$

cannot be obtained from an inner product since this norm does not satisfy the parallelogram equality (4). Indeed, if we take $x(t) = 1$ and

$y(t) = (t - a)/(b - a)$, we have $\|x\| = 1$, $\|y\| = 1$ and

$$x(t) + y(t) = 1 + \frac{(t - a)}{(b - a)}$$

$$x(t) - y(t) = 1 - \frac{(t - a)}{(b - a)}$$

Hence $\|x + y\| = 2 \; and \; \|x - y\| = 1$

$\|x + y\|^2 + \|x - y\|^2 = 5 \text{but} 2(\|x\|^2 + \|y\|^2) = 4$

This completes the proof.

It is remarkable that, conversely, we can "rediscover" the inner product from the corresponding norm. In fact, the reader may verify by straightforward calculation that for a real inner product space we have

(12) $$\langle x, y \rangle = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2)$$

and for a complex inner product space we ha

(13) $$Re\langle x, y \rangle = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2)$$

$$Im\langle x, y \rangle = \frac{1}{4}(\|x + iy\|^2 - \|x - iy\|^2)$$

Formula (13) is sometimes called the polarization identity.

Example 7: The space $L^2[a, b]$, the space of all square integrable functions over $[a, b]$ is a Hilbert space.

Solutions:-Define the inner product on $L^2[a, b]$ by $\langle x, y \rangle = \int_a^b \left| x(t)\overline{y(t)} \right| \, dt, \forall \, x, y \in L^2[a, b]$ and the norm on $L^2[a, b]$is given by $||x|| = \sqrt{\int_z^b |x(t)|^2 \, dt}$. Also with respect to this norm it can be shown that $L^2[a, b]$ is complete with respect to a metric defined by

$d(x, y) = \left[\int_a^b |x(t) - y(t)|^2\right]^{1/2}$So $L^2[a, b]$] is a Hilbert space.

Remark:

Whenever we are addressed with the problem of verifying whether given function defines a norm or not, the first three properties will be more or less obvious, and most of the effort, if any, would go in verifying the triangle inequality.

So, once a vector space with a norm would be called a normed linear space. So, given a normed linear space we can define a metric $d(x,y) = \|x-y\| \geq 0, \forall x,y \in X$ It is clear that $d(x,y)$is non-negative and $d(x,y) = 0$if andonly if $x = y$. Now, by the triangle inequality, we get $d(x,y) \leq d(x,z) + d(z,y); \forall x,y,z \in X$Therefore, the distancefunction d satisfies the usual triangle inequality for a metric; and that is why we have the same name for these two inequalities.

Therefore, automatically a normed linear space gets a topology defined by this norm which is a nice metric topology; and that is called the norm topology of this vector space.

SOME NORMS

S.No.	**Space**	**Norm** $\|\boldsymbol{x}\|$
1.	$\mathbb{R}^n$ and $\mathbb{C}^n$	$\left(\sum_{j=1}^{n} \lvert x_j \rvert^2\right)^{1/2} = \sqrt{\lvert x_1\rvert^2 + \cdots \lvert x_n\rvert^2}$
2.	l^p	$\left(\sum_{j=1}^{\infty} \lvert x_j \rvert^p\right)^{1/p}$ where $1 \leq p < \infty$
3.	l^∞	$sup_j \lvert x_j \rvert$ if $p = \infty$
4.	$C[a,b]$	$max_{t \in j} \lvert x(t) \rvert$
5.	Set of all continuous real –valued functions on [0,1]	$\int_0^1 \lvert x(t) \rvert \, dt$

Note:

$l^p \subset l^{p'}$ if $l \leq p \leq p' \leq \infty$.

Note:

$c = \{x \in l^\infty : \big(x(j)\big)$converges in $\mathbb{K}\}$.

$c_0 = \{x \in c : \big(x(j)\big)$converges to 0 in $\mathbb{K}\}$.

$$c_{00} = \left\{x \in l^p \; all \; but \; finetely \; many x_j' sare \; 0\right\}, 1 \leq p \leq \infty.$$

Note:

For $1 \leq p < \infty$,by $L^p(E)$, we mean a collection of equivalence classes $[f]$ for which $|f|^p$ is integrable. Thus

$$f \in L^p(E) \Leftrightarrow \int_E |f|^p < \infty.$$

Sometimes we denote the collection of such functions by the symbol L^p.

Note:

A measurable function f on measurable set E is said to be an essentially bounded function if there exists $M_f > 0$ such that

$|f(x)| \leq M_f$for all most all $x \in E$.

We define $L^\infty(E)$to be the collection equivalence classes $[f]$ for which $f$is essentially bounded functions on E.

Therefore $f \in L^\infty(E) \Leftrightarrow$ there exists $M_f > 0$ such that $|f(x)| \leq M_f$for almost all $x \in E$.

Note:

For E a measurable set, $1 \leq p < \infty$, and a function f in $L^p(E)$, we denote

$\|f\|_p \coloneqq (\int_E |f|^p)^{1/p}$, and for $p = \infty, \|f\|_\infty = inf \{M_f > 0 :$

$|f(x)| \leq M_f$for almost all $x \in E\}$.

Note:

For $1 \leq p \leq \infty$, $L^p(E)$ is a vector space over$R$.

8.5 SUMMARY

This unit we have start from some basic definitions (inner product space, Hilbert space). After that we have defined these spacesthe properties then result and examples defined.

8.6 GLOSSARY

i. **Inner Product Space:** A function$\langle\cdot,\cdot\rangle$that maps pairs of vectors to real or complex numbers, satisfying conjugate symmetry, linearity, and positivity.

ii. **Norm:** A function $\|\cdot\|$ derived from the inner product, defined as$\| x \| = \langle x, x\rangle$representing the length or magnitude of a vector.

iii. **Orthogonality:** A condition where two vectors x, y are orthogonal if $\langle x, y\rangle = 0$

iv. **Hilbert Space**: A complete inner product space, meaning every Cauchy sequence converges to a limit within the space.

v. **Completeness:** A property of a space where every Cauchy sequence has a limit that is also within the space.

vi. **Cauchy Sequence:** A sequence $\{x_n\} where\ for\ every\ \epsilon > 0$, there exists an$N$ such that for all $m, n > N$, $\| x_m - x_n \| < \epsilon$.

vii. **Orthonormal Set:** A set of vectors that are orthogonal to each other and each have unit norm$(\| x \| = 1)$

viii. **Orthonormal Basis:** A basis consisting of orthonormal vectors, which allows for straightforward vector decomposition and reconstruction.

ix. **Cauchy-Schwarz Inequality:**An inequality stating

$$| \langle x, y\rangle | \leq \| x \| \| y \| \text{ for all vectors } x \text{ and } y.$$

x. **Triangle Inequality**:An inequality stating

$$\| x + y \| \leq \| x \| + \| y \| \text{for all vectors } x \text{ and y.}$$

xi. **L^2 Space:** The space of square-integrable functions, where the inner product is defined by $\langle f, g\rangle = \int f(x)\overline{g(x)}\, dx$.

xii. **Finite-Dimensional Hilbert Space:** Any finite-dimensional inner product space, which is automatically complete and thus a Hilbert space.

CHECK YOUR PROGRESS

Fill in the Blanks:

1. A vector space equipped with an inner product is called an __________

2. A __________ is a complete inner product space.

3. In an inner product space, two vectors are said to be __________ if their inner product is zero.

4. The inner product of a vector with itself is always __________ and is zero if and only if the vector is the __________ vector.

5. In a Hilbert space, every __________ sequence converges to a limit within the space.

6. The __________-Schwarz inequality is a fundamental property of inner product spaces.

7. The space of square-integrable functions, denoted by __________, is an example of a Hilbert space.

True/False

8. Every Hilbert space is an inner product space..True/False.

9. Every inner product space is a Hilbert space. True/False.

10. In a Hilbert space, every Cauchy sequence converges to a limit within the space.True/False.

11. The norm defined by an inner product always satisfies the triangle inequality.True/False.

12. Orthogonal vectors in an inner product space always have an inner product of zero.True/False.

13. All finite-dimensional inner product spaces are Hilbert spaces.True/False.

14. The inner product of two vectors in an inner product space is always a real number. True/False.

15. A complete normed space is known as a : Hilbert space

- **i.** Compact space
- **ii.** Banach space
- **iii.** Euclidean space
- **iv.** Hilbert space

16. The term Hilbert space stands for a :

- **i.** Complete inner product space
- **ii.** Compact linear space
- **iii.** Complete normed space
- **iv.** Complete metric space

17. Which of the following is Cauchy-Schwartz inequality?

(a) $|\langle x, y\rangle| \leq \langle x, x\rangle^{1/2} \cdot \langle y, y\rangle^{1/2}$

(b) $|\langle x, y\rangle| \geq \langle x, x\rangle^{1/2} \cdot \langle y, y\rangle^{1/2}$

(c) $|\langle x, y\rangle| \leq \langle x, y\rangle^{1/2} \cdot \langle y, x\rangle^{1/2}$

(d) $|\langle x, y\rangle| \leq \langle x, x\rangle \cdot \langle y, y\rangle$

18. The distance between any two orthonormal vectors in an inner product space is:

a) 1

b) $\sqrt{2}$

c) 1

d) 2

19. Which of the following is known as the parallelogram law?

(a) $\|x+y\|^2 = 2\|x\|^2 + 2\|y\|^2$

(b) $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + \|y\|^2$

(c) $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$

(d) $\|x+y\|^2 - \|x-y\|^2 = 2\|x\|^2 + \|y\|^2$

20. Two vectors x, y in an inner product space are orthogonal if :

(a) $\langle x, y\rangle = 0$

(b) $\|x\| = \|y\| = 1$

(c) $\langle x, y\rangle \neq 0$

(d) None of these.

21. If two vectors x, y in an inner product space are orthogonal, then:

(a) $\|x+y\|^2 = 2\|x\|^2 + 2\|y\|^2$

(b) $\|x+y\|^2 = \|x\|^2 + \|y\|^2$

(c) $\|x+y\| = 0$

(d) None of these.

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8.9 TERMINAL QUESTIONS

1. If an inner product space X is real, show that the condition

$$\|x\| = \|y\| \text{implies} \langle x + y, x - y \rangle = 0.$$

What does this mean geometrically if $X = \mathbb{R}^2$? What does the condition imply if X is complex?

What is an inner product space, and how is it defined.

2. Explain the concept of a Hilbert space. What makes a Hilbert space different from a general inner product space?
3. What is orthogonality in the context of inner product spaces? Provide an example.
4. Provide an example of a Hilbert space that is not finite-dimensional.

8.10 ANSWERS

CHECK YOUR PROGRESS

a. Inner product space.

b. Hilbert space

c. Orthogonal

d. Non-negative, Zero

e. Cauchy

f. Cauchy

g. L^2

h. True

i. False

j. True

k. True

l. True

m. True

n. False

o. (ii)

p. (i)

q. (a)

r. (b)

s. (c)

t. (a)

u. (b)

UNIT 9:

PRPERTIES OF INNER PRODUCT SPACE

CONTENTS:

9.1 INTRODUCTION

After completion of previous unit the learner are familiar about inner product space. Now in the present unit we are explaining the properties of inner product space.

9.2 OBJECTIVES

After studying this unit, learner will be able to

i. Described the concept of Isomorphism of an inner product space.

ii. Explained the Orthogonal Complements and Direct Sums.

iii. Describe the idea of an Orthonormal Sets and Sequence.

9.3 LEMMA AND THEOREM

Lemma 1 (Schwarz inequality, triangle inequality).

An inner product and the corresponding norm satisfy the Schwarz inequality and the triangle inequality as follows.

(a) $|< x, y >| \leq \|x\|\|y\|$ (Schwarz inequality)

...(1)

where the equality sign holds if and only if $\{x, y\}$ is a linearly dependent set.

(b) That norm also satisfies

$\|x + y\| \leq \|x\| + \|y\|$ (Triangle inequality).

....(2)

Where the equality sign holds if and only if $y = 0$ or $x = cy$ (c real and ≥ 0).

Proof.

(a) If $y = 0$, then (1) holds since $< x, 0 \geq 0$.

Let $y \neq 0$.

For every scalar α we have,

$$0 \leqq \|x - \alpha y\|^2 = \langle x - \alpha y, x - \alpha y\rangle$$

$$= \langle x, x\rangle - \bar{\alpha}\langle x, y\rangle - \alpha[\langle y, x\rangle - \bar{\alpha}\langle y, y\rangle].$$

We observe that the expression in the brackets [……..] is zero if we choose,

$$\bar{\alpha} = \langle y, x\rangle / \langle y, y\rangle.$$

The remaining inequality is,

$$0 \leqq \langle x, x\rangle - \frac{\langle y, x\rangle}{\langle y, y\rangle}\langle x, y\rangle = \|x\|^2 - \frac{|\langle x, y\rangle|^2}{\|y\|^2};$$

We are using here

$$\langle y, x\rangle = \overline{\langle x, y\rangle}.$$

Multiplying by $\|y\|^2$, transferring the last term to the left and taking square roots, we obtain (1).

Equality holds in this derivation if and only if $y = 0 \; or \; 0 = \|x - \alpha y\|^2$ hence $x - \alpha y = 0$, so that $x = \alpha y$, which shows linear dependence.

(b) We prove (2). We have,

$$\|x+y\|^2 = \langle x+y,\ x+y\rangle = \|x\|^2 + \langle x,\ y\rangle + \langle y,\ x\rangle + \|y\|^2.$$

By the Schwarz inequality,

$$|\langle x,\ y\rangle| = |\langle y,\ x\rangle| \leqq \|x\|\ \|y\|.$$

By the triangle inequality for numbers we thus obtain,

$$\begin{aligned} \|x+y\|^2 &\leqq \|x\|^2 + 2\ |\langle x,\ y\rangle| + \|y\|^2 \\ &\leqq \|x\|^2 + 2\ \|x\|\ \|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2. \end{aligned}$$

Taking square roots on both sides, we have (2). Equality holds in this derivation if and only if,

$$\langle x,\ y\rangle + \langle y,\ x\rangle = 2\ \|x\|\ \|y\|.$$

The left-hand side is $2Re < x, y >$, where Re denotes the real part. From this and (1),

$$\text{(3)} \qquad \operatorname{Re}\langle x,\ y\rangle = \|x\|\ \|y\| \geqq |\langle x,\ y\rangle|.$$

Since the real part of a complex number cannot exceed the absolute value, we must have equality, which implies linear dependence by part (a), say $y = 0\ or\ x = cy$. We show that c is real and ≥ 0. From (3) with the equality sign we have

$$\operatorname{Re}\langle x,\ y\rangle = |\langle x,\ y\rangle|.$$

But if the real part of a complex number equals the absolute value, the imaginary part must be zero.

Hence $\langle x, y\rangle = \mathrm{Re}\langle x, y\rangle \geqq 0$ by (3), and $c \geqq 0$ follows from

$$0 \leqq \langle x, y\rangle = \langle cy, y\rangle = c\,\|y\|^2.$$

Lemma 2.(Continuity of inner product).

If in an inner product space ,

$$x_n \longrightarrow x \textit{ and } y_n \longrightarrow y, \textit{ then } \langle x_n, y_n\rangle \longrightarrow \langle x, y\rangle.$$

Proof. Subtracting and adding a term, using the triangle inequality for numbers and, finally, the Schwarz inequality, we obtain

$$\begin{aligned}|\langle x_n, y_n\rangle - \langle x, y\rangle| &= |\langle x_n, y_n\rangle - \langle x_n, y\rangle + \langle x_n, y\rangle - \langle x, y\rangle| \\ &\leqq |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle| \\ &\leqq \|x_n\|\,\|y_n - y\| + \|x_n - x\|\,\|y\| \longrightarrow 0\end{aligned}$$

Since,

$$y_n - y \longrightarrow 0 \text{ and } x_n - x \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

9.4 ISOMORPHISM OF AN INNER PRODUCT SPACE

An isomorphism T of an inner product space X onto an inner product space $\hat{X}$ over the same field is a bijective linear operator $T: X \to \hat{X}$ Which preserves the inner product, that is, for all $x, y \in X$,

$$< Tx, Ty > = < x, y >,$$

where we denoted inner products on X and $\hat{X}$ by the same symbol, for simplicity. $\hat{X}$ is then called isomorphic with X and X and $\hat{X}$ are called isomorphic inner product spaces.

9.5 THEOREM

Theorem 3. For any inner product space X there exists a Hilbert space H and an isomorphism A from X onto a dense subspace $\subset H$. The space H is unique except for isomorphisms

Proof. Since we know the result, if $\mathrm{X} = (\mathrm{X}, \|.\|)$ be a normed space. Then there is a Banach space $\widehat{\mathrm{X}}$ and an isometry A from X onto a subspace W of X which is dense in $\widehat{\mathrm{X}}$. The space $\widehat{\mathrm{X}}$ is unique, except for isometries.

For reasons of continuity, under such an isometry, sums and scalar multiples of elements in X and W correspond to each other, so that A is even an isomorphism of X onto W, both regarded as normed spaces.

Lemma 2 shows that we can define an inner product on H by setting,

$$\langle \hat{x}, \hat{y} \rangle = \lim_{n \to \infty} \langle x_n, y_n \rangle,$$

The $\langle x_n \rangle$ and (y_n) are representatives of $\hat{x} \in H$ and $\hat{y} \in H$, respectively.

From previous unit we see that A is an isomorphism of X onto W, both regarded as inner product spaces. In starting lines we also explain the guarantees that H is unique except for isometries, that is, two completions H and $\widehat{H}$ of X are related by an isometry $T: H \longrightarrow \widehat{H}$. Reasoning as in the case of A, we conclude that T must be an isomorphism of the Hilbert space H onto the Hilbert space $\widehat{H}$.

Subspace of an inner product space:

A subspace Y of an inner product space X is defined to be a vector subspace of X taken with the inner product on X restricted to $Y \times Y$.

Similarly, a subspace Y of a Hilbert space H is defined to be a subspace of H, regarded as an inner product space. Note that Y need not be a Hilbert space because Y may not be complete.

Theorem 4:

Let Y be a subspace of a Hilbert space H. Then:

(a) *Y is complete if and only if Y is closed in H.*

(b) *If Y is finite dimensional, then Y is complete.*

(c) *If H is separable, so is Y. More generally, every subset of a separable inner product space is separable.*

9.6 ORTHOGONAL COMPLEMENTS AND DIRECT SUMS

In a metric space X, the *distance* δ from an element $x \in X$ to a nonempty subset $M \subset X$ is defined to be

$$\delta = \inf_{\tilde{y} \in M} d(x, \tilde{y}) \qquad (M \neq \emptyset).$$

In a normed space this becomes

(1) $$\delta = \inf_{\tilde{y} \in M} \|x - \tilde{y}\| \qquad (M \neq \emptyset).$$

(2) $$\delta = \|x - y\|,$$

where, $y \in M$

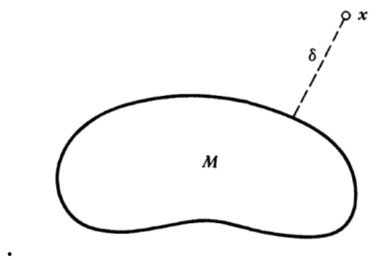

Fig.9.5.1

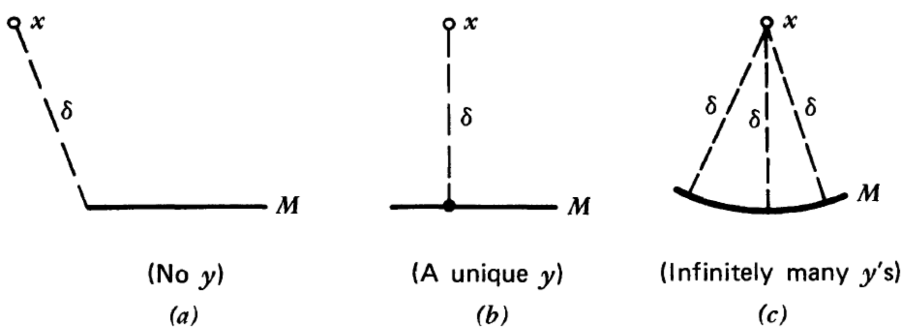

Fig 9.5.2

Existence and uniqueness of points $y \in M$ satisfying (2), where the given $M \subset \mathbf{R}^2$ is an open segment [in (a) and (b)] and a circular arc [in (c)]

The above figure illustrates that even in a very simple space such as the Euclidean plane R^2 there may be no y satisfying (2) or precisely one such y, or more than one y.

And we may expect that other spaces, in particular infinite dimensional ones, will be much more complicated in that respect. For general normed spaces this is the case but for Hilbert spaces the situation remains relatively simple. This fact is surprising and has various theoretical and practical consequences. It is one of the main reasons why the theory of Hilbert spaces is simpler than that of general Banach spaces.

9.6.1 SEGMENT AND CONVEX SET

To consider that existence and uniqueness problem for Hilbert spaces and to formulate the below results, we need two related concepts, which are of general interest, as follows.

The segment joining two given elements x and y of a vector space X is defined to be the set of all $z \in X$ of the form

$$z = \alpha x + (1-\alpha) y \qquad (\alpha \in \mathbf{R}, 0 \leqq \alpha \leqq 1).$$

A subset M of X is said to be convex if for every $x, y \in M$the segment joining x and y is contained in M.

For instance, every subspace Y of X is convex, and the intersection of convex sets is a convex set.

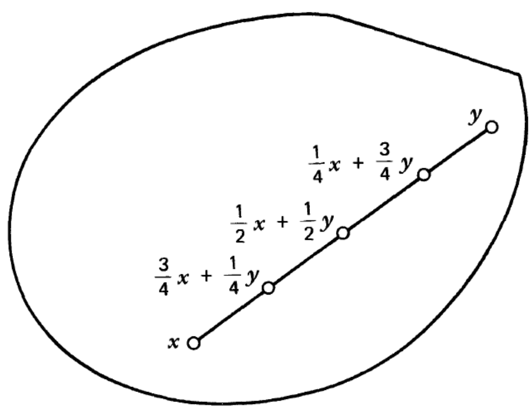

Fig.9.5.1

Segment in a convex set

9.6.2 DIRECT SUM

A vector space X is said to be the *direct sum* of two subspaces Y and Z of X, written

$$X = Y \oplus Z,$$

if each $x \in X$ has a unique representation

$$x = y + z$$

$y \in Y,\ z \in Z.$

Then Z is called an algebraic complement of Y in X and vice versa, and Y, Z is called a complementary pair of subspaces in X.

For example, $Y = \mathbf{R}$ is a subspace of the Euclidean plane $\mathbf{R}^2$. Clearly, Y has infinitely many algebraic complements in $\mathbf{R}^2$, each of which is a real line. But most convenient is a complement that is perpendicular. We make use of this fact when we choose a Cartesian coordinate system. In $\mathbf{R}^3$ the situation is the same in principle.

Similarly, in the case of a general Hilbert space H, the main interest concerns representations of H as a direct sum of a closed subspace Y and its **orthogonal complement**

$$Y^{\perp} = \{z \in H \mid z \perp Y\},$$

which is the set of all vectors orthogonal to Y.

9.6.3 LEMMA AND THEOREM

Theorem 5:

Let X be an inner product space and $M \neq \varnothing$ a convex subset which is complete (in the metric induced by the inner product). Then for every given $x \in X$ there exists a unique $y \in M$ such that

$$(3) \qquad \delta = \inf_{\tilde{y} \in M} \|x - \tilde{y}\| = \|x - y\|.$$

Proof. **(a)** *Existence.* By the definition of an infimum there is a sequence (y_n) in M such that

$$(4) \qquad \delta_n \longrightarrow \delta \qquad \text{where} \qquad \delta_n = \|x - y_n\|.$$

We show that (y_n) is Cauchy. Writing $y_n - x = v_n$, we have $\|v_n\| = \delta_n$ and

$$\|v_n + v_m\| = \|y_n + y_m - 2x\| = 2\,\|\tfrac{1}{2}(y_n + y_m) - x\| \geqq 2\delta$$

because M is convex, so that $\frac{1}{2}(y_n + y_m) \in M$. Furthermore, we have $y_n - y_m = v_n - v_m$. Hence by the parallelogram equality,

$$\|y_n - y_m\|^2 = \|v_n - v_m\|^2 = -\|v_n + v_m\|^2 + 2(\|v_n\|^2 + \|v_m\|^2)$$
$$\leqq -(2\delta)^2 + 2(\delta_n{}^2 + \delta_m{}^2),$$

and (4) implies that (y_n) is Cauchy. Since M is complete, (y_n) converges, say, $y_n \longrightarrow y \in M$. Since $y \in M$, we have $\|x - y\| \geqq \delta$.

Also by (4),

$$\|x - y\| \leqq \|x - y_n\| + \|y_n - y\| = \delta_n + \|y_n - y\| \longrightarrow \delta.$$

This shows that $\|x - y\| = \delta$.

(b) *Uniqueness.* We assume that $y \in M$ and $y_0 \in M$ both satisfy

$$\|x - y\| = \delta \qquad \text{and} \qquad \|x - y_0\| = \delta$$

and show that then $y_0 = y$. By the parallelogram equality,

$$\|y - y_0\|^2 = \|(y - x) - (y_0 - x)\|^2$$

$$= 2\,\|y - x\|^2 + 2\,\|y_0 - x\|^2 - \|(y - x) + (y_0 - x)\|^2$$

$$= 2\delta^2 + 2\delta^2 - 2^2\,\|\tfrac{1}{2}(y + y_0) - x\|^2.$$

On the right, $\frac{1}{2}(y + y_0) \in M$, so that

$$\|\tfrac{1}{2}(y + y_0) - x\| \geqq \delta.$$

This implies that the right-hand side is less than or equal to $2\delta^2 + 2\delta^2 - 4\delta^2 = 0$. Hence we have the inequality $\|y - y_0\| \leqq 0$. Clearly, $\|y - y_0\| \geqq 0$, so that we must have equality, and $y_0 = y$.

In from above theorem following lemma can be proved.

Lemma 3.

let M be a complete subspace Y and $x \in X$ fixed. Then $z = x - y$ is orthogonal to Y.

Proof. If $z \perp Y$ were false, there would be a $y_1 \in Y$ such that

$$\langle z, y_1 \rangle = \beta \neq 0. \tag{5}$$

Clearly, $y_1 \neq 0$ since otherwise $\langle z, y_1 \rangle = 0$. Furthermore, for any scalar α,

$$\|z-\alpha y_1\|^2 = \langle z-\alpha y_1, z-\alpha y_1\rangle$$
$$= \langle z, z\rangle - \bar{\alpha}\langle z, y_1\rangle - \alpha[\langle y_1, z\rangle - \bar{\alpha}\langle y_1, y_1\rangle]$$
$$= \langle z, z\rangle - \bar{\alpha}\beta - \alpha[\bar{\beta} - \bar{\alpha}\langle y_1, y_1\rangle].$$

The expression in the brackets $[\cdots]$ is zero if we choose

$$\bar{\alpha} = \frac{\bar{\beta}}{\langle y_1, y_1\rangle}.$$

From (3) we have $\|z\| = \|x-y\| = \delta$, so that our equation now yields

$$\|z-\alpha y_1\|^2 = \|z\|^2 - \frac{|\beta|^2}{\langle y_1, y_1\rangle} < \delta^2.$$

But this is impossible because we have

$$z - \alpha y_1 = x - y_2 \qquad \text{where} \qquad y_2 = y + \alpha y_1 \in Y,$$

so that $\|z-\alpha y_1\| \geqq \delta$ by the definition of δ. Hence (5) cannot hold, and the lemma is proved.

Theorem 6:

Let Y be any closed subspace of a Hilbert space H. Then

$$(6) \qquad H = Y \oplus Z \qquad Z = Y^{\perp}.$$

Lemma 4:

The orthogonal complement $Y^{\perp}$ of a closed subspace Y of a Hilbert space H is the null space $\mathcal{N}(P)$ of the orthogonal projection P of H onto Y.

An orthogonal complement is a special annihilator, where, by definition, the *annihilator* $M^{\perp}$ of a set $M \neq \varnothing$ in an inner product space X is the set

$$M^{\perp} = \{x \in X \mid x \perp M\}.$$

Thus, $x \in M^{\perp}$ if and only if $\langle x, v\rangle = 0$ for all $v \in M$.

$(M^{\perp})^{\perp}$ is written $M^{\perp\perp}$, etc. In general we have

(8*) $$M \subset M^{\perp\perp}$$

Lemma 5:

If Y is a closed subspace of a Hilbert space H, then

(8) $$Y = Y^{\perp\perp}.$$

Lemma 6:

For any subset $M \neq \varnothing$ of a Hilbert space H, the span of M is dense in H if and only if $M^{\perp} = \{0\}$.

9.7 ORTHONORMAL SETS AND SEQUENCES

An orthogonal set M in an inner product space X is a subset $M \subset X$ whose elements are pairwise orthogonal. An *orthonormal set* $M \subset X$ is an orthogonal set in X whose elements have norm 1, that is, for all $x, y \in M$,

(1) $$\langle x, y\rangle = \begin{cases} 0 & \text{if } x \neq y \\ 1 & \text{if } x = y. \end{cases}$$

If an orthogonal or orthonormal set M is countable, we can arrange it in a sequence (x_n) and call it an *orthogonal* or *orthonormal sequence*, respectively.

More generally, an indexed set, or *family*, (x_α), $\alpha \in I$, is called *orthogonal* if $x_\alpha \perp x_\beta$ for all $\alpha, \beta \in I$, $\alpha \neq \beta$. The family is called *orthonormal* if it is orthogonal and all x_α have norm 1, so that for all $\alpha, \beta \in I$ we have

$$\langle x_\alpha, x_\beta \rangle = \delta_{\alpha\beta} = \begin{cases} 0 & \text{if } \alpha \neq \beta, \\ 1 & \text{if } \alpha = \beta. \end{cases} \tag{2}$$

Here, $\delta_{\alpha\beta}$ is the Kronecker delta

In particular, we may take the family defined by the *natural injection* of M into X, that is, the restriction to M of the identity mapping $x \longmapsto x$ on X.

For orthogonal elements x, y we have $\langle x, y \rangle = 0$, so that we readily obtain the **Pythagorean relation**

(3) $$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

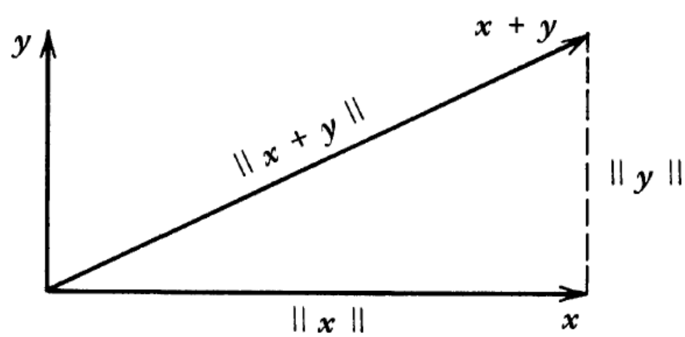

Fig.9.7.1

Pythagorean relation (3) in $\mathbf{R}^2$

More generally, if $\{x_1, \cdots, x_n\}$ is an orthogonal set, then

$$\|x_1 + \cdots + x_n\|^2 = \|x_1\|^2 + \cdots + \|x_n\|^2. \tag{4}$$

In fact, $\langle x_j, x_k \rangle = 0$ if $j \neq k$; consequently,

$$\left\| \sum_j x_j \right\|^2 = \left\langle \sum_j x_j, \sum_k x_k \right\rangle = \sum_j \sum_k \langle x_j, x_k \rangle = \sum_j \langle x_j, x_j \rangle = \sum_j \|x_j\|^2$$

Lemma 7: An orthonormal set is linearly independent.

Proof. Let $\{e_1, \cdots, e_n\}$ be orthonormal and consider the equation

$$\alpha_1 e_1 + \cdots + \alpha_n e_n = 0.$$

Multiplication by a fixed e_j gives

$$\left\langle \sum_k \alpha_k e_k, e_j \right\rangle = \sum_k \alpha_k \langle e_k, e_j \rangle = \alpha_j \langle e_j, e_j \rangle = \alpha_j = 0$$

and proves linear independence for any finite orthonormal set. This also implies linear independence if the given orthonormal set is infinite, by the definition of linear independence.

9.8 SUMMARY

In starting of the unit we have given some Lemma and Theorem then Isomorphism of an inner product space is defined. After that Orthogonal Complements and Direct Sums: Segment and Convex Set, Direct Sum, Orthogonal complement and Orthonormal Sets and Sequence defined in a simple manner.

9.9 GLOSSARY

i. **Metric space:** Let $X \neq \emptyset$ be a set then the metric on the set X is defined as a function $d: X \times X \to [0, \infty)$ such that some conditions are satisfied.

ii. **Vector space: -** Let V be a nonempty set with two operations

(i) **Vector addition:** If any $u, v \in V$ then $u + v \in V$

(ii) **Scalar Multiplication:** If any $u \in V$ and $k \in F$ then $ku \in V$

Then V is called a vector space (over the field F) if the following axioms hold for any vectors if the some conditions hold.

iii. **Normed space:-** Let X be a vector space over scalar field K. A *norm* on a (real or complex) vector space X is a real-valued function on X ($\|x\|: X \to K$) whose value at an $x \in X$ is denoted by $\|x\|$ and which has the four properties here x and y are arbitrary vectors in X and α is any scalar.

iv. **Banach space:-** A complete normed linear space is called a Banach space.

v. Inner product space.

vi. Hilbert space.

CHECK YOUR PROGRESS

Fill in the blanks

1. If in an ....,

$$x_n \longrightarrow x \text{ and } y_n \longrightarrow y, \text{ then } \langle x_n, y_n \rangle \longrightarrow \langle x, y \rangle.$$

2. A subset M of X is said to be ....if for every $x, y \in M$the segment joining x and y is contained in M.

3. *For any subset $M \neq \emptyset$ of a Hilbert space*

   H, the span of M is dense in H

   ....

4. An isomorphism T of an inner product space X onto an inner product space $\hat{X}$ over the same field is a bijective linear operator $T: X \to \hat{X}$ Which preserves the ....

5. For any inner product space X there exists a Hilbert space H and an isomorphism A from X onto a............. The space H is unique except for isomorphisms

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9.12 TERMINAL QUESTIONS

1.

What is the Schwarz inequality in $\mathbf{R}^2$ or $\mathbf{R}^3$? Give another proof of it in these cases.

2.

Give examples of subspaces of l^2.

3.

Show that in an inner product space, $x \perp y$ if and only if $\|x + \alpha y\| \geqq \|x\|$ for all scalars α.

4. Give examples of representations of $\mathbb{R}^3$ as a direct sum (i) of a subspace and its orthogonal complement, (ii) of any complementary pair of subspaces.

9.13 ANSWERS

CHECK YOUR PROGRESS

1. inner product space
2. convex
3. *if and only if* $M^{\perp} = \{\dot{0}\}$.
4. inner product
5. dense subspace $\subset H$

UNIT 10: HILBERT ADJOINT OPERATOR AND OTHER OPERATORS

CONTENTS:

10.1 INTRODUCTION

Definition10.1.1. Let $T: H_1 \to H_2$ be a bounded linear operator, where H_1andH_2 are Hilbert spaces. Thenthe Hilbert-adjoint operator T^* of T is the operatorsuch thatfor all $x \in H_1$ and $y \in H_2$,

$$\langle Tx, y\rangle = \langle x, T^*y\rangle$$

We should show that this definition makes sense, that is, for a given T, T^* does exist and it is unique. Before this, consider the following example and a remark:

Example 10.1.1. Consider: $H_1 = H_2 = R^4, x = (x_1, x_2, x_{3,} x_{4,})$,

$y = (y_1, y_2, y_{3,} y_{4,})$ and$T: R^4 \to R^4$ is defined as

$$T(x_1, x_2, x_{3,} x_{4,}) = (x_2, x_3, x_{4,}, 0).$$

Then,

$$\langle x, T^* y\rangle = \langle Tx, y\rangle$$

$$\Rightarrow \langle x, T^* y\rangle = \langle T(x_1, x_2, x_{3,} x_{4,}), (y_1, y_2, y_{3,} y_{4,})\rangle$$

$$= \langle (x_2, x_3, x_{4,}, 0), (y_1, y_2, y_{3,} y_{4,})\rangle$$

$$= x_2 y_1 + x_3 y_2 + x_4 y_3$$

$$= \langle (x_1, x_2, x_{3,} x_{4,}), (0, y_1, y_2, y_3)\rangle$$

Therefore, $T^* y \left(= \ T^*(y_1, y_2, y_{3,} y_{4,})\right) = (0, y_1, y_2, y_3)$.

Remark 10.1.1. Fora bounded linear operator$T: H_1 \to H_2$, its double Hilbert adjoint is same as T, i.e. $T^{**} = T$.

Proof: For each $x \in H_1$ and $y \in H_2$, $\langle Tx, y\rangle = \langle x, T^* y\rangle = \overline{\langle T^* y, x\rangle} = \overline{\langle y, T^{**} x\rangle} = \langle T^{**} x, y\rangle$. Thus, $T = T^{**}$.

Theorem 10.1.1 The Hilbert adjoint operator T^* of a bounded linear operator T exists, unique and is bounded linear operator with norm $||T|| = ||T^*||$.

Proof: For given $y \in H_2$, $h_y(x) = \langle Tx, y\rangle$, for each $x \in H_1$,defines a linear functional onH_1. This can be seen as follows:

$$h_y(\alpha x + \beta z) = \langle T(\alpha x + \beta z), y\rangle$$

$$= \langle \alpha Tx + \beta Tz, y\rangle$$

$$= \langle \alpha Tx, y\rangle + \langle \beta Tz, y\rangle$$

$$= \alpha\langle Tx, y\rangle + \beta\langle Tz, y\rangle$$

$$= \alpha h_y(x) + \beta h_y(z).$$

Thus, $h_y(x)$ is a linear functional in first quadrant. Now by the Schwarz inequality

$$|h_y(x)| = |Tx, y| \leq ||Tx|| \times ||y|| \leq ||T|| \times ||x|| \times ||y||.$$

Hence, $||h_y|| \leq ||T|| \times ||y||$.

Now, by Reisz representation theorem there exists a unique $y_0 \in H_1$ such that

$h_y(x) = \langle x, y_0 \rangle$ and $||h_y|| = ||y_0||$.

This implies that $\langle Tx, y \rangle = \langle x, y_0 \rangle$.

Define, $T^*: H_2 \to H_1$ as $T^*y = y_0$. Hence for the operator T, its Hilbert adjoint exists and it is unique.

Also, $||T^*y|| = ||y_0|| = ||h_y|| \leq ||T|| \times ||y||$.

This implies that $||T^*|| \leq ||T||$. And, since $T^{**} = T$, therefore $||T^{**}|| \leq ||T^*||$i.e. $||T|| \leq ||T^*||$. Hence $||T^*|| = ||T||$.

Proposition 10.1.1 Let X and Y be inner product spaces and $S: X \to Y$ be a bounded linear operator. Then:

a) $S = 0$ if and only if $\langle Sx, y \rangle = 0$ for all $x \in X$ and $y \in Y$.
b) If $S: X \to X$, where X is over complex field, then $\langle Sx, x \rangle = 0$for all $x \in X$ if and only if $S = 0$.

Proof: (a) If $S = 0$, then $Sx = 0$ for all $x \in X$. And hence $\langle Sx, y \rangle = 0$ for all $x \in X$ and $y \in Y$.

Conversely, assume that $\langle Sx, y \rangle = 0$ for all $x \in X$ and $y \in Y$. Then putting $y = Sx$, we have $\langle Sx, Sx \rangle = 0$ for all $x \in X$. This implies that $||Sx||^2 = 0$, for all $x \in X$. Consequently, $Sx = 0$, for all $x \in X$.

(b) It is very obvious that if $S = 0$, then $\langle Sx, x \rangle = 0$for all $x \in X$. Conversely, assume that if $\langle Sx, x \rangle = 0$for all $x \in X$, then by polarization identity $\langle Sx, y \rangle = 0$, for all $x, y \in X$. And thus by part (a) $S = 0$.

Remark 10.1.2. Note that, in the statement (b) of above proposition X is over complex field is essential. In case of, X is over real field, this result may not be true. For e.g. take $X = R^2$ and $S: R^2 \to R^2$ defined as $S(x_1, x_2) = (-x_2, x_1)$. Then in this case $\langle Sx, x\rangle = 0$, whereas $S \neq 0$.

10.2PROPERTIES OF HILBERT ADJOINT OPERATOR

Theorem 10.2.1 Let H_1, H_2be Hilbert spaces, $S: H_1 \to H_2$ and $T: H_1 \to H_2$ bounded linear operators and $\propto$ be any scalar. Then we have

- **a)** $\langle T^*y, x\rangle = \langle y, Tx\rangle$
- **b)** $(S+T)^* = S^* + T^*$
- **c)** $(\propto T)^* = \bar{\propto} T^*$
- **d)** $(T^*)^* = T$
- **e)** $\|T^*T\| = \|TT^*\| = \|T\|^2$
- **f)** $T^*T = 0$ if and only if $T = 0$
- **g)** $(ST)^* = T^*S^*(assuming\ H_2 = H_1)$, and hence$(T^n)^* = (T^*)^n$

Proof. (a)For all $x \in H_1$ and $y \in H_2$,

$$\langle T^*y, x\rangle = \overline{\langle x, T^*y\rangle} = \overline{\langle Tx, y\rangle} = \langle y, Tx\rangle.$$

(b) For all $x \in H_1$ and $y \in H_2$,

$$\langle x, (S+T)^*y\rangle = \langle (S+T)x, y\rangle$$
$$= \langle Sx, y\rangle + \langle Tx, y\rangle$$
$$= \langle x, S^*y\rangle + \langle x, T^*y\rangle$$
$$= \langle x, (S^* + T^*)y\rangle.$$

Hence $(S+T)^*y = (S^* + T^*)y$ for all $y \in H_2$. And hence $(S+T)^* = S^* + T^*$.

(c) $\langle (aT)^*y, x\rangle = \langle y, (aT)x\rangle$

$$= \langle y, a(Tx)\rangle$$

$$= \bar{a}\langle y, Tx\rangle$$
$$= \bar{a}\langle T^*y, x\rangle$$
$$= \langle \bar{a}T^*y, x\rangle$$

(d)This part is already done in the remark.

(e) The operatorsT^*T: $H_1 \rightarrow H_{1,}$ but $TT^*H_2 \rightarrow H_2$, By the Schwarz inequality.

$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle \leq \|T^*Tx\|\|x\| \leq \|T^*T\|\|x\|^2$.

Taking the supremum over all x, we obtain $\|T\|^2 \leq \|T^*T\|$.

And

$$\|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2.$$

Hence $\|T^*T\| = \|T\|^2$. Replacing T by T^*in previous equation we get $\|TT^*\| = \|T\|^2$.

(f) Using the above result (e) $\|T^*T\| = \|TT^*\| = \|T\|^2$, we get $T^*T = 0$ if and only if $T = 0$ if and only if $TT^* = 0$.

(g) Assume that, $H_2 = H_1 = H$, Since, For all $x \in H$ and $y \in H$,

$\langle x, (ST)^*y\rangle = \langle (ST)x, y\rangle = \langle Tx, S^*y\rangle = \langle x, T^*S^*y\rangle$.

Hence$(ST)^* = T^*S^*$.

10.3 SELF-ADJOINT, NORMAL AND UNITARY OPERATOR

Definition 10.3.1. A bounded linear operator $T: H \rightarrow H$ on a Hilbert space H is said to be**self adjoint** if $T^* = T$.

Definition 10.3.2. A bounded linear operator $T: H \rightarrow H$ on a Hilbert space H is said to be**normal** if $TT^* = T^*T$.

Definition 10.3.3. A bounded linear operator $T: H \to H$ on a Hilbert space H is said to be**unitary** if$I$ is bijective and T*= T^{-1}, i.e. $TT^* = T^*T = I$.

Remark 10.3.1: The Hilbert-adjoint operator T* of T is defined by,$\langle Tx, y\rangle = \langle x, T * y\rangle$. Then:

i. If T is self-adjoint, we see that the formula become$\langle Tx, y\rangle = \langle x, Ty\rangle$.

ii. If T is self-adjoint, then T is normal.

iii. If T is unitary, then T is normal.

Exercise 10.3.1: Give an example of an operator T such that:

i. T is normal, but not unitary.

ii. T is normal, but not self adjoint.

iii. T is self adjoint, but not unitary.

Theorem 10.3.1 (Self-adjointness). Let $T: H \to H$ be a bounded linearoperator on a Hilbert space H. Then:

a. If T is self-adjoint, $\langle Tx, x\rangle$ is real for all $x \in H$.

b. If H is over complex field and $\langle Tx, x\rangle$ is real for all $x \in H$, the operator T is self-adjoint.

Proof. **(a)**IfT is self-adjoint, then for all x,

$$\overline{\langle Tx, x\rangle} = \langle x, Tx\rangle = \langle Tx, x\rangle,$$

Hence$\langle Tx, x\rangle$ is equal to its complex conjugate, so that it is real.

(b) if $\langle Tx, x\rangle$ is real for all x, then

$$\langle Tx, x\rangle = \overline{\langle Tx, x\rangle} = \overline{\langle xT^*, x\rangle} = \langle T^*x, x\rangle.$$

Hence,

$$0 = \langle Tx, x\rangle - \langle T^*x, x\rangle = \langle (T - T^*)x, x\rangle$$

Andsince H is over complex field, therefore$T - T^* = 0$. And thus $T = T^*$, i.e. T is self adjoint.

Remark 10.3.2: (i) In the statement (b) of the above theorem it is essential that H is over complex field. This is very clear that this statement may not be true if H is overreal field.Since for a real H the inner product is real-valued, which makes $\langle Tx, x\rangle$ real always, regardless T is zero operator or not.

(ii) Products (composites) of self-adjoint operators appear quite often in applications, so that the following theorem will be useful.

10.4 IMPORTANT THEOREMS ON SELF-ADJOINT, NORMAL AND UNITARY OPERATOR

Theorem 10.4.1 (Self-adjointness of product). The product of two bounded self-adjoint linear operators S and T on a Hilbert space H is self-adjoint if and only if the operators commute.

$$ST = TS.$$

Proof. We have already proven that,

$$(ST)^* = T^*S^* = TS.$$

Hence, from the above equation $ST = (ST)^*$is true if and only if $ST = TS$.

Proposition 10.4.1: (i) If T is a self adjoint operator on a Hilbert space H then T^n is also self adjoint for every $n \geq 1$.

(ii) If T is a self adjoint operatoron a Hilbert space H, then $\left||T^n|\right| = \left||T|\right|^n$ for every $n \geq 1$.

Proof: (i)We know that $(T^n)^* = (T^*)^n$, and given that $T = T^*$. Therefore, $(T^n)^* = T^n$. Hence, T^n is self adjoint for every $n \geq 1$.

(ii) We know that for any bounded linear operator T, $||T||^2 = ||TT^*||$.

And if T is self adjoint, then $||T||^2 = ||T^2||$.

Also $||T^4|| = ||T^2||^2 = ||T||^4$. Similarly for any $2^k, k \geq 1$, we have $\left\|T^{2^k}\right\| = \left\|T^{2^{k-1}}\right\|^2 = \left\|T^{2^{k-2}}\right\|^4 = \cdots = ||T^4||^{2^{k-2}} = ||T^2||^{2^{k-1}} = ||T||^{2^k}$. Thus, $\left\|T^{2^k}\right\| = ||T||^{2^k}$ for any $2^k, k \geq 1$.

Now, for any $1 \leq n \leq 2^k$,

$$\left\|T^{2^k}\right\| = \left\|T^n T^{2^k-n}\right\| \leq ||T^n|| \times \left\|T^{2^k-n}\right\| \leq ||T^n|| \times ||T||^{2^k-n}$$

$$\leq ||T||^n \times ||T||^{2^k-n} = ||T||^{2^k}.$$

In the above inequality the first and last term are equal therefore all the term in between are also equal.

Thus, 4th and 5th term are also equal, and this gives $||T^n|| = ||T||^n$ for every $n \geq 1$.

Theorem 10.4.2: Let H be a Hilbert space over complex field. Then every bounded linear operator T on H can be represented as $T = T_1 + iT_2$, where T_1andT_2 are self adjoint operator. And this representation is unique.

Proof: Define$T_1 = \frac{1}{2}(T + T^*)$ and $T_2 = -\frac{1}{2}i(T - T^*)$. Then $T_1^* = \frac{1}{2}(T^* + T)$ and $T_2^* = \frac{1}{2}i(T^* - T)$. Therefore $T_1^* = T_1$ and $T_2^* = T_2$. Hence T_1andT_2 are self adjoint operator. Also from the definition of T_1andT_2, $T_1 + iT_2 = T$.

Uniqueness: Let $T = S_1 + iS_2$ be another representation such that $S_1$and S_2 are self-adjoint. Then

$$T_1 + iT_2 = S_1 + iS_2.$$

This implies that

$$(T_1 - S_1) + i(T_2 - S_2) = 0.$$

Also, zero operator is self adjoint operator, therefore

$$[(T_1 - S_1) + i(T_2 - S_2)] = [(T_1 - S_1) + i(T_2 - S_2)]^*$$

$$= (T_1 - S_1)^* - i(T_2 - S_2)^* = (T_1 - S_1) - i(T_2 - S_2).$$

This gives $T_2 - S_2 = 0$, i.e. $T_2 = S_2$. Consequently using the second last equation, we get $T_1 = S_1$. Hence, representation is unique.

Theorem 10.4.3. Let H be Hilbert space over complex field and T be a bounded linear operator, then

(a) T is normal iff $\|Tx\| = \|T^*x\|$ for all $x \in H$.

(b) If T is normal then $K(T) = K(T^*)$

(c) If T is normal then $\|T^n\| = \|T\|^n$ for every $n \geq 1$.

Proof. (a) Clearly,

$$\|Tx\|^2 - \|T^*x\|^2 = 0$$

$$\Leftrightarrow \langle Tx, Tx\rangle - \langle T^*x, T^*x\rangle = 0$$

$$\Leftrightarrow \langle T^*Tx, x\rangle - \langle TT^*x, x\rangle = 0$$

$$\Leftrightarrow \langle (T^*T - TT^*)x, x\rangle = 0$$

Since H is over complex field, therefore $T^*T - TT^* = 0$ if and only if $\langle (T^*T - TT^*)x, x\rangle = 0$ for every $x \in H$.

(b) If T is normal then, by (a) $\|Tx\| = \|T^*x\|$. This gives, $Tx = 0$ if and only if $T^*x = 0$. Hence If T is normal then, $K(T) = K(T^*)$.

(c) Remember the property (e) of Hilbert adjoint operator which is $||T||^2 = ||T^*T||$. Also, it is easy to check that for any operator T, T^*T is self adjoint, therefore,

$$||(T^*T)^n|| = ||TT^*||^n.$$

Now, consider T is normal then $(T^*T)^n = (T^*)^n T^n$. Then using these facts, we have the following:

$$\|T\|^{2n} = \|T^*T\|^n = \|(T^*T)^n\| = \|(T^*)^nT^n\|$$
$$\leq \|(T^*)^n\| \times \|T^n\|$$
$$\leq \|T^*\|^n \times \|T^n\|$$
$$\leq \|T^*\|^n \times \|T\|^n$$
$$= \|T\|^n \times \|T\|^n$$
$$= \|T\|^{2n}.$$

In the above inequalities first and last term are same, therefore all the in between terms are equal. Therefore, $\|T^n\| = \|T\|^n$ for every $n \geq 1$.

Theorem 10.4.4 Let$H$ be a Hilbert space and the operators $U\colon\ H \to H$ and $V\colon\ H \to H$ be unitary. Then:

(a) U is isometric, thus $\|Ux\| = \|x\|$ for all $x \in H$

(b) $\|U\| = 1$, provided$H \neq 0$,

(c) $U^{-1}(= U^*)$ is unitary,

(d) UV is unitary,

(e) U is normal.

Furthermore:

(a) A bounded linear operator T on a complex Hilbert space H is unitary if and only if T is isometric and surjective.

Proof. **(a)** For $x \in H$,

$$\|Ux\|^2 = \langle Ux, Ux\rangle = \langle x, U^*Ux\rangle = \langle x, Ix\rangle = \|x\|^2.$$

This implies that U is an isometry.

(b) Since, U is unitary, therefore $\|Ux\| = \|x\|$. This immediately follows that $\|U\| = 1$.

(c) Since U is bijective, so is U^{-1}, and by $U^{-1} = U^*$, we have

$$(U^{-1})^* = U^{**} = U.$$

Therefore$(U^{-1})^* U^{-1} = UU^{-1} = I = U^{-1}U = U^{-1}(U^{-1})^*$. This follows that $U^{-1}$is also unitary.

(d) Since U and V are bijective, therefore $UV$is bijective, and

$$(UV)^* = V^*U^* = V^{-1}U^{-1} = (UV)^{-1}.$$

This implies that UV is unitary.

(e) Follows from definition.

(f) Suppose that T is isometric and surjective. Isometry implies T is one-one, so that T is bijective. We show that $T^* = T^{-1}$. By the isometry,

$$\langle T^*Tx, x\rangle = \langle Tx, Tx\rangle = \langle x, x\rangle = \langle Ix, x\rangle.$$

Hence

$$\langle (T^*T - I)x, x\rangle = 0$$

and $T^*T - I = 0$(H isover complex field), so that $T^*T = I$. From this,

$$TT^* = TT^*(TT^{-1}) = T(T^*T)T^{-1} = TIT^{-1} = I.$$

Together, $T^*T = TT^* = I$. Hence $T^* = T^{-1}$, so that T is unitary. Conversely, suppose that T is isometric and surjective, therefore by definition$T$ is unitary.

Remark 10.4.1 Note that an isometric operator need not be unitary since it may fail to be surjective. An example is the right shift operator $T\colon\ l^2 \to l^2$ given by

$$T\left(x_1, x_2, x_{3,} \ldots\ldots\right) = (0, x_1, x_2, \ldots\ldots)$$

where $\left(x_1, x_2, x_{3,} \ldots\ldots\right) \in l^2$. This operator is isometry but not unitary.

10. 5SOLVED EXAMPLES

Question 10.5.1: Show that (a) $0^* = 0$ and (b)$I^* = I$.

Solution:

a. $\langle x, 0^*y\rangle = \langle 0x, y\rangle$

$= \langle 0, y\rangle = 0$

$= \langle x, 0y\rangle$

This implies that $0^* = 0$.

b. $\langle x, I^*y\rangle = \langle Ix, y\rangle$

$= \langle x, y\rangle$

$= \langle x, Iy\rangle$

This implies that $I^* = I$.

Question 10.5.2: Suppose λ is a eigen value of T. Is it true that$\bar{\lambda}$ is an eigen value for T^*.

Solution: Not true. Consider $T: l^2 \to l^2$ as

$$T(x_1, x_2, x_{3,} \ldots\ldots) = (x_2, x_3, \ldots\ldots)$$

Let λ be eigen value of T, Then :

$$T(x_1, x_2, x_{3,} \ldots\ldots) = \lambda(x_1, x_2, x_3, \ldots\ldots)$$

$$\Rightarrow (x_2, x_3, \ldots\ldots) = (\lambda x_1, \lambda x_2, \lambda x_3, \ldots\ldots)$$

$\lambda = 0$ is a eigen value for T, and its corresponding eigen vector is $(x_1, 0,0,0,0 \ldots\ldots)$.

For non-zero λ: $x_2 = \lambda x_3, x_3 = \lambda x_4, x_4 = \lambda x_5, x_5 = \lambda x_6,$ and so on. Eigen vector must be non-zero, therefore $x_i \neq 0$, for all i, because if

$x_i = 0$, for some i, then $x_{i+1} = 0 = x_{i+2} = \cdots$ and also $x_{i-1} = 0 = x_{i-2} = \cdots = x_1$.

Since $T(x_1, x_2, x_{3,} \ldots\ldots) \in l^2$, therefore in this case $(\lambda x_1, \lambda x_2, \lambda x_3, \ldots\ldots) \in l^2$.

$$\Rightarrow (\lambda x_1, \lambda^2 x_1, \lambda^3 x_1, \ldots\ldots) \in l^2$$

$$\Rightarrow \sum_{i=1}^{\infty} \left|\lambda^i x_1\right|^2 < \infty$$

$$\Rightarrow |x_1|^2 \sum_{i=1}^{\infty} \left|\lambda^i\right|^2 < \infty$$

$$\Rightarrow \sum_{i=1}^{\infty} \left|\lambda^i\right|^2 < \infty$$

$$\Rightarrow |\lambda|^2 < 1$$

$$\Rightarrow |\lambda| < 1$$

Thus eigenvalue for T is $\{\lambda: |\lambda| < 1\}$. Now it is easy to see that Hilbert adjoint of T is

$$T^*(x_1, x_2, x_{3,} \ldots\ldots) = (0, x_1, x_2, \ldots\ldots)$$

For eigen value λ of T^*:

$$T^*(x_1, x_2, x_{3,} \ldots\ldots) = \lambda(x_1, x_2, \ldots\ldots)$$

$$\Rightarrow (0, x_1, x_2, \ldots\ldots) = (\lambda x_1, \lambda x_2, \lambda x_3, \ldots\ldots)$$

This implies that, $\lambda x_1 = 0, \lambda x_2 = x_1, \lambda x_3 = x_2, \ldots..$ and so on. If $\lambda = 0$, then $x_1 = 0, x_2 = 0, x_3 = 0, \ldots$. And if $\lambda \neq 0$, then

$$\lambda x_1 = 0 \Rightarrow x_1 = 0,$$

$$\lambda x_2 = x_1 \Rightarrow x_2 = 0,$$

similarly $x_3 = 0 = x_4 = \cdots$. Thus T^* has no eigen value.

Question 10.5.3: If T is self adjoint operator, then all eigen values are real.

Solution: Let λ be eigen value and $x \neq 0$ be corresponding eigen vector. Then

$$\langle Tx, x\rangle = \langle x, T^*x\rangle$$
$$\Rightarrow \langle \lambda x, x\rangle = \langle x, Tx\rangle$$
$$= \langle x, \lambda x\rangle$$
$$\Rightarrow \lambda\langle x, x\rangle = \bar{\lambda}\langle x, x\rangle$$
$$\Rightarrow \lambda = \bar{\lambda}.$$

Hence λ is real.

Question 10.5.4: Give an example of an operator T such that T^*T is identity operator but TT^* is not an identity operator.

Solution: Consider the right shift operator T: $l^2 \to l^2$ given by

$$T(x_1, x_2, x_{3,} \ldots\ldots) = (0, x_1, x_2, \ldots\ldots)$$

Its adjoint operator is

$$T^*(x_1, x_2, x_{3,} \ldots\ldots) = (x_2, x_3, x_4 \ldots\ldots).$$

Then

$$T^*T(x_1, x_2, x_{3,} \ldots\ldots) = (x_1, x_2, x_{3,} \ldots\ldots)$$

Whereas,

$$TT^*(x_1, x_2, x_{3,} \ldots\ldots) = (0, x_2, x_3, x_4 \ldots\ldots).$$

10.6 SUMMARY

After the learning of this unit, the students are able to:

- **i.** Understand the concept of Hilbert adjoint operator
- **ii.** Analyse a relation between kernel of T and range of T^*.
- **iii.** Analyse the idea of self adjoint operator
- **iv.** Analyse the idea of normal operator

v. Analyse the idea of unitary operator.

10.7 GLOSSARY

i. Hilbert adjoint operator

ii. Kernel of T

iii. Kernel of T^*

iv. Range of T

v. Range of T^*

vi. Self adjoint operator

vii. Normal Operator

viii. Unitary Operator

10.8 TERMINAL QUESTIONS

TQ 10.8.1 If $T: H \rightarrow H$ is bounded linear operator, then show that $\overline{R(T)} = K(T^*)^{\perp}$, where, $R(T)$ is range of T and $K(T)$ is kernel of T.

TQ 10.8.2 If $T: H \rightarrow H$ is bounded linear operator, then show that $K(T) = R(T^*)^{\perp}$, where, $R(T)$ is range of T and $K(T)$ is kernel of T.

TQ 10.8.3 Let H be a Hilbert space and let U be a bounded linear operator such that $R(U) = H$. Then show that the following are equivalent:

(a) U is unitary.

(b) U is an isometry: $\|Ux\| = \|x\|$ for every $x \in H$;

(c) U preserves the inner product: $\langle Ux, Uy \rangle = \langle Tx, Tx \rangle$ for all $x, y \in H$.

TQ 10.8.4 Let H be a Hilbert space and $T: H \rightarrow H$ be a bijective bounded linear operator whose inverse is bounded. Show that $(T^*)^{-1}$ exists and $(T^*)^{-1} = (T^{-1})^*$.

10.9ANSWER TO TERMINAL QUESTION

TQ 10.8.1 Let $y \in R(T)$. Then, there exists $x \in H$ such that $Tx = y$.
Then, for any $z \in K(T^*)$, $\langle y, z\rangle = \langle Tx, z\rangle = \langle x, T^*z\rangle = 0$. This implies that $y \in K(T^*)^{\perp}$. $R(T) \subseteq K(T^*)^{\perp}$.
Now, let $x \in R(T)^{\perp}$. Then for all $y \in H, \langle Ty, x\rangle = 0 = \langle y, T^*x\rangle$. This implies that $T^*x = 0$. And hence, $x \in K(T^*)$. Therefore, $R(T)^{\perp} \subseteq K(T^*)$.
This further implies that $K(T^*)^{\perp} \subseteq R(T)^{\perp\perp} = \overline{R(T)}$. Hence $\overline{R(T)} = K(T^*)^{\perp}$.

TQ 10.8.2 Since, $T^{**} = T$ and using TQ 10.8.1 on T^* instead of T, we have $\overline{R(T^*)}^{\perp} = K(T^{**})^{\perp\perp}$. This implies that $K(T) = R(T^*)^{\perp}$.

TQ 10.8.3 : See theorem 10.4.4

TQ 10.8.4 See the definition of self-adjoint operator and use the concept of bijectivity.

BLOCK IV: FUNDAMENTAL THEOREMS FOR NORMED AND BANACH SPACES

UNIT 11:

HAHN-BANACH THEOREM

CONTENTS:

11.1 INTRODUCTION

The Hahn-Banach theorem is a cornerstone of functional analysis, a branch of mathematical analysis that studies vector spaces endowed with a topology, typically infinite-dimensional. Named after Hans Hahn and Stefan Banach, who independently formulated the theorem in the early 20th century, this result has profound implications in both pure and applied mathematics.The development of the Hahn-Banach theorem marks a pivotal moment in the evolution of functional analysis. During the early 1900s, mathematicians were focused on generalizing classical results from finite-dimensional vector spaces to infinite-dimensional contexts. The theorem's origin can be traced back to Hahn's work in 1927, which was later extended by Banach in 1929.

The Hahn Banach theorem is a central tool in functional analysis. It allows the extension of bounded linear functionals defined a vector subspace of some vector space to the whole space, and it also shows that there are "enough"continuous linear functionals defined on every normed vector space to make the study of the dual space "interesting". Another version of the Hahn–Banach theorem is known as the **Hahn–Banach separation theorem** or the hyperplane separation theorem and has numerous uses in convex geometry.The Hahn–Banach theorem arose from attempts to solve infinite systems of linear equations. This is needed to solve problems such as the moment problem, whereby given all the potential moment of a function one must determine if a function having these moments exists, and, if so, find it in terms of those moments. Another such problem is the Fourier cosine series problem, whereby given all the potential Fourier cosine coefficients one must determine if a function having those coefficients exists, and, again, find it if so.

11.2 OBJECTIVES

After studying this unit, learner will be able to

i. **Understand the Statement and Proof** Comprehend the formal statement of the Hahn-Banach Theorem and its proof, including the key concepts and techniques used.

ii. **Extend Linear Functionals**: Apply the Hahn-Banach Theorem to extend linear functionals from a subspace of a vector space to the whole space while preserving their norm.

iii. **Dual Spaces**: Understand the concept of dual spaces and how the Hahn-Banach Theorem ensures the richness of the dual space by guaranteeing the existence of many continuous linear functionals.

iv. **Functional Analysis Applications**: Apply the theorem in various problems and proofs in functional analysis, including in the study of weak topologies, reflexivity, and the representation of dual spaces.

v. **Problem Solving**: Solve advanced problems in functional analysis and related fields using the Hahn-Banach Theorem as a tool.

11.3 BASICS

We first defined the basic definitions:

11.3.1 PARTIAL ORDER SET

Partial Order Set: Let X be a non-empty set a relation R on X is said to be a partial order if

- **i.** R is reflexive (i.e. x R x, $\forall$ x $\in$ X.)
- **ii.** R is anti-symmetric i.e. x R y, y R x, $\Longrightarrow$ x = y, $\forall$ x, y $\in$ X.
- **iii.** R is transitive i.e. x R y, y R z $\Longrightarrow$ x R z, $\forall$ x, y, z $\in$ X.

Note:

- Let R be partial order relation on X and (x, y) $\in$ R. Then we write x $\leq$ y.
- A non-empty set X with a partial order relation $\leq$ defined on X i.e. (X, $\leq$) is called a partial order set.

Definition: Let (X, $\leq$) be a partial order set then

- **i.** An element a $\in$ X is said to be an upper bound of X if x $\leq$ a, $\forall$ x $\in$ X.
- **ii.** An element α $\in$ X is said to be a least upper bound of X if α is an upper bound of X and a is an upper bound of X then α $\leq$ a.
- **iii.** An element x $\in$ X is said to be a maximal element of X if x $\leq$ y, y $\in$ X $\Longrightarrow$ x=y.
- **iv.** Let A be a subset of X. Then set A is said to be linearly order set or chain if x, y $\in$ A, $\Longrightarrow$ x $\leq$ y or y $\leq$ x.

Example

i. Let N be a natural number and $n \leq m$ if $\frac{n}{m}$, A = {2, 4, 5} them L.U.B. = 20.

ii. N = {1, 2, 3, …, 50, 51, …, 100} and $m \geq n$ if $\frac{n}{m}$, then maximal elements are 51, 52, 53,…,100.

11.3.2 ZORN'S LEMMA

Zorn's Lemma- If every chain in a partial order set $(X, \leq)$ has an upper bound then there is a maximal element in X.

Sub linear functional: Let X be a vector space over a filed R. A function p on X into R is said to be sub linear function if

i. $p(x+y) \leq p(x) + p(y)$,$\forall$ x, y $\in$ X.

ii. $P(\alpha x) = \alpha\, p(x)$, $\alpha \in R$, $x \in X$.

Exercise: Let X be a vector space over a filed R and Y be a proper linear subspace of X, let g be a linear functional on Y. let $x_0 \in X \setminus Y$, define a function G on $Y \oplus [x_0]$ by

$G(y + \alpha x_0) = g(y) + \alpha C \quad \forall\ y + \alpha x_0 \in Y \oplus [x_0]$, Where C is fixed. Then G is a linear functional on $Y \oplus [x_0]$

11.4 HAHN BANACH THEOREM FOR REAL VECTOR SPACE

Hahn-Banach Theorem for real vector space- Let X be a vector space over the field of real line and p be a sub linear functional on X defined as

$$p(x+y) \leq p(x) + p(y),\ \forall\ x, y \in X.$$

$$P(\alpha x) = \alpha\ p(x),\ \alpha \in R,\ x \in X.$$

Let M be a linear sub space of X, let f be a linear functional on M such that

$$f(x) \leq p(x) \quad \forall\ x \in M.$$

Then there is a linear functional F on X such that

$$F(x) = f(x)$$

$$F(x) \leq p(x)\ \forall\ x \in X.$$

For the proof of this theorem, a lemma is required.

Lemma: Let X be a vector space over a field R and p be a sub linear functional on X. Let Y be a proper subspace of X, let g be a linear functional on Y such that

$$g(y) \leq p(y) \quad \forall\ y \in Y.$$

Let $x_0 \in X \setminus Y$. Then there is a linear functional G on $Y \oplus [x_0]$ into R such that

$$G(y) = g(y) \quad \forall\ y \in Y,$$

$$G(y + \alpha x_0) \leq p(y + \alpha x_0) \quad \forall\ y + \alpha x_0 \in Y \oplus [x]$$

Proof of Hahn Banach theorem: Proof is an application of Zorn's lemma. Let Ƒ be a collection of linear functional g from X into R such that

i. $M \subseteq D(g)$

ii. $g(x) = f(x) \quad \forall\ x \in M$

iii. $g(x) \leq p(x)\ \forall\ x \in M$

And $G(y + \alpha x_0) \leq p(y + \alpha x_0)$ $\forall \alpha \in R$, for g, h ∈ Ƒ if

$D(g) \subseteq D(h)$

$g(x) = h(x)\ \forall\ x \in D(g)$.

It is easy to see that ≤ is a partial order relation on Ƒ. Then (Ƒ, ≤) is a partial order set. Let

$C = \{g_\alpha : \alpha \in \Delta\}$ be a chain in Ƒ. Let $= \cup_{\alpha \in \Delta} D(g_\alpha)$

Since $C = \{g_\alpha : \alpha \in \Delta\}$ be a chain in Ƒ, let $g_\alpha, g_\beta \in C$, therefore either $g_\alpha \leq g_\beta$ or $g_\alpha \geq g_\beta$ that implies that $D(g_\alpha) \subseteq D(g_\beta)$ or $D(g_\alpha) \supseteq D(g_\beta)$, let x ∈ D and x belong to $D(g_\alpha)$ as well as $D(g_\beta)$ then

$g_\alpha(x) = g_\beta(x)$ for such x

It follows that D is a linear subspace of X, define g on D into R by

$$g(x) = g_\alpha(x)\ \forall\ x \in D(g_\alpha).$$

Then g is well defined on D, and g is linear, also $M \subseteq D(g) = \cup_{\alpha \in \Delta} D(g_\alpha)$ = D

$(x) = g_\alpha(x) = f(x)\ \forall\ x \in M$ (C ⊆ Ƒ)

$g(x) = g_\alpha(x) \leq p(x)\ \forall\ x \in \cup_{\alpha \in \Delta} D(g_\alpha) = D$ then g ∈ Ƒ

Next $g_\alpha \leq g\ \forall \alpha \in \Delta$, ($D(g_\alpha) \subseteq D(g) = \cup_{\alpha \in \Delta} D(g_\alpha) = D$

Since g is upper bound of the chain C in Ƒ. Then by Zorn's lemma (Ƒ , ≤) has a maximal element in Ƒ. Let F be the maximal element in (Ƒ , ≤). Now we claim that D(F) = X proof by contradiction, suppose D(F) ≠ X, let $x_0 \in X \setminus D(F)$. Then by using the Lamma, there exists a linear functional G on $D(F) \oplus [x_0]$ into R, defined by

$G(x) = F(x)\ \forall\ x \in D(F)$

And $G(x) \leq p(x)\ \forall\ x \in D(F) \oplus [x_0]$

Clearly $M \subseteq D(G)$ $(D(F) \subseteq D(F) \oplus [x_0])$

$G(x) = F(x) = f(x)\ \forall\ x \in M$

$G(x) = F(x) \leq p(x)\ \forall\ x \in D(G)$

Then G ∈ Ƒ and F ≤ G (D(F) ⊆ D(G))

Also F ≠ G

This is a contradiction the maximal of F, F is the required extension of f

$$F(x) = f(x)\ \forall\ x \in M$$

And $$F(x) \le p(x)\ \ \forall\ x \in X.$$

Corollary:

Let X be a vector space over a field K(R). Let p be a semi-norm on X, let M be a linear subspace of X. Let f be linear functional on M into R such that $|f(x)| \le p(x)\ \forall\ x \in M$. Then there is a linear functional F on X into R such that

- **i.** $F(x) = f(x)\ \ \forall\ x \in M$
- **ii.** $|f(x)| \le p(x)\ \ \forall\ x \in X$.

Proof.

We need that a semi norm in a sub linear functional then

$f(x) \le |f(x)| \le p(x)\ \ \forall\ x \in M$, then $f(x) \le p(x)\ \ \forall\ x \in M$. By Hahn Banach theorem, for real vector space, there exists a linear functional F on X into R such that

$$F(x) = f(x)\ \forall\ x \in M$$

And $$F(x) \le p(x)\ \ \forall\ x \in X$$

Since $F(x) \le p(x)\ \ \forall\ x \in X$ and F is a linear functional on X

$$F(-x) \le p((-1)x)\ \ \ \forall\ x \in X$$

$$-F(x) \le |-1|\, p(x)\ \ \ \forall\ x \in X$$

$$-F(x) \le p(x)\ \ \forall\ x \in X$$

$$|F(x)| \le p(x)\ \ \forall\ x \in X.$$

11.4.1 HAHN BANACH THEOREM FOR COMPLEX VECTOR SPACE

Hahn Banach theorem for Complex vector spaces:

Let X be vector space over a field of complex number and p a semi norm on X. Let M be a linear subspace of X. Let f be linear functional on M into C such that

$$|f(x)| \leq p(x) \quad \forall\, x \in M.$$

Then there is a linear functional f on X into C such that

$$F(x) = f(x) \quad \forall\, x \in M.$$

$$|F(x)| \leq p(x) \quad \forall\, x \in X.$$

Proof. Define u on M into R by

$$u(x) = \text{Real part } (f(x)) \quad \forall\, x \in M$$

Then $u(x+y) = \text{Re } (f(x + iy)$

$u(x+y) = \text{Re } (f(x) + f(y))$ (since f is linear)

$u(x+y) = \text{Re } (f(x)) + \text{Re } f(y))$

$u(x+y) = u(x) + u(y) \quad \forall\, x, y \in M$

and $\alpha \in R$, $u(\alpha x) = \text{Re } (f(\alpha(x))$

$u(\alpha x) = \text{Re } \alpha f((x)$ (since f is linear)

$u(\alpha x) = \alpha \text{ Re } f(x) = \alpha\, u(x)$.

Since u : M → R is a real linear functional, Also $|u(x)| = \text{Re } (f(x)) \leq |f(x)|$ $\forall\, x \in M$

This implies that $|u(x)| \leq p(x)\ \forall\, x \in M$. Now by Hahn Banach theorem for real vector space there exists a real linear functional U on X into r such that

$$U(x) = u(x)\ \forall\, x \in M$$

$$U(x) \leq p(x)\ \forall\, x \in M$$

$$U(x) = u(x)\ \forall\, x \in M$$

Define F : X → C by

$F(x) = U(-i^2(x+y)) – iU(i(x + y))$

$F(x) = U(-i^2x + -i^2y) – i U(ix + iy)$

$F(x) = U (x+y) – i U (ix + iy)$

$F(x) = U(x) + U(y) – i(U(ix) + U(iy))$ (U is linear)

$F(x) = F(x) + F(y)$

For all $\alpha \in R$, $F(\alpha x) = U((\alpha x) – iU(i\alpha x)$

$F(\alpha x) = \alpha U(x) – i\alpha U(ix)$

$F (\alpha x) = \alpha (U(x) – i U(ix)$

$F(\alpha x) = \alpha F(x)$ For all $\alpha \in R$, $x \in X$.

Also $F(ix) = U(ix) – i U (i^2 x)$

$F(ix) = U(ix) – i U(-x)$

$F(ix) = U(ix) + iU(x)$

$F(ix) = iF(x)$ For all $x \in X$.

Let $C = \alpha + i\beta$, $\alpha, \beta \in R$

$F(c(x)) = F((\alpha + i\beta)x)$

$= F(\alpha x + i\beta x) = F(\alpha x) + F(i\beta x)$

$= \alpha F(x) + \beta F(ix) = \alpha F(x) + i\beta F(x)$

$= (\alpha + i\beta) F(x) = c F(x)$

$F(c(x)) = c F(x)$

$F : X(C) \to C$ is a linear functional now

$F(x) = U(x) – iU(ix)$

$F(x) = Re(f(x)) – iRe(f(ix))$ Far all $x \in M$

$F(x) = Re(f(x)) – iRe(if(x))$ Far all $x \in M$

$F(x) = Re(f(x)) + iIm(f(x))$ Far all $x \in M$

$F(x) = f(x)$ Far all $x \in M$

It is remained to show that $|f(x)| \leq p(x)$ Far all $x \in X$

$F(x) = |F(x)| e^{i\theta}$, θ is real

$|F(x)| = F(x) e^{-i\theta}$

$|F(x)| = F(x e^{-i\theta})$

$|F(x)| = | U (x e^{-i\theta}) |$

(since |F(x)| is real number ≥ 0)

$|F(x)| \leq p\,(x\, e^{-i\theta})$ ($|U(x)| \leq p(x)$ For all x ∈ X)

$|F(x)| \leq |\, e^{-i\theta}|\, p(x)$ { p is semi linear}

$|F(x)| \leq p(x)$ For all x ∈ X.

11.4.2 HAHN BANACH THEOREM FOR NORMED LINEAR SPACE

Hahn Banach theorem for Normed linear space: Let (X, || ||) be a normed linear space over a filed K (=R or C) and M be a linear subspace of X. Let f be a bounded linear functional on M. Then there is a bounded linear functional F on X such that

- **i.** $F(x) = f(x) \ \forall\, x \in M$
- **ii.** $||F|| = ||f||$

Proof. Define $p : X \to R$ by

$$P(x) = ||f||\, ||x|| \quad \forall\, x \in X$$

It is easy to see that p is semi norm on X

$$|f(x)| \leq ||f||\, ||x|| \quad \forall\, x \in M$$

$$|f(x)| \leq p(x) \quad \forall\, x \in M$$

By Hahn Banach theorem , there is a linear functional F on X such that

$$F(x) = f(x) \quad \forall\, x \in M$$

And $$|F(x)| \leq p(x) \quad \forall\, x \in X$$

$$|F(x)| \leq ||f||\, ||x|| \quad \forall\, x \in X$$

$$||F|| \leq ||f||$$

Next $$||f|| = sup_{x \in M} \frac{|f(x)|}{||x||}$$

$$||f|| = sup_{x \in M} \frac{|F(x)|}{||x||} \quad (F(x) = f(x)\ \forall\, x \in M)$$

$$||f|| \leq sup_{x \in M} \frac{|F(x)|}{||x||} = ||F||$$

$$||f|| \leq ||F||$$

Hence ||F|| = ||f||.

11.4.3 APPLICATION OF HAHN BANACH THEOREM

1. Let (X, || ||) be a normed linear space over a filed K (=R or C). Let X be a non-zero vector in x. then there is a bounded linear functional F on X such that

i. F(x) = ‖x‖

ii. ||F‖ = 1.

Proof Given x ∈X and x ≠ 0, Let M = [x] = {α x : α ∈ K}

Define f : M → K by

$$f(\alpha x) = \alpha \|x\| \quad \forall \alpha \in K$$

Let x ∈ M and α_1, α_2∈ K

$f(\alpha_1 x + \alpha_2 x) = f((\alpha_1+\alpha_2)x) = \alpha_1+\alpha_2 \|x\|$

$f(\alpha_1 x + \alpha_2 x) = \alpha_1 \|x\| + \alpha_2 \|x\|$

$= f(\alpha_1 x) + f(\alpha_2 x)$

For all β ∈K, $f(\beta(\alpha x)) = f(\beta\alpha x) = \alpha\beta \|x\|$

$f(\beta(\alpha x)) = \beta(\alpha \|x\|)$

$f(\beta(\alpha x)) = \beta f(\alpha x)$

Since f in linear functional on M then $f(\alpha x) = \alpha \|x\| \quad \forall x \in K$

Let α = 1 then $F(x) = \|x\|$

$f(\alpha x) = \alpha \|x\| \quad \forall x \in K$

$\|f(\alpha x)\| = |\alpha| \|x\| = \|\alpha x\|$

Hence $\| f \| \leq 1$

f is a bounded linear functional on M then

$$\| f \| = sup_{\alpha \in K} \frac{|f(\alpha x)|}{|| \alpha x ||}$$

$$\| f \| = sup_{\alpha \in K} \frac{|\alpha| \, ||x||}{|\alpha| \, || x ||} = 1$$

Hence $\| f \| = 1$.

By Hahn Banach theorem there is a bounded linear functional F on X such that

$F(\alpha x) = f(\alpha x) \ \forall x \in K$

And $\| F \| = \| f \| = 1$

In particular taking $\alpha = 1$ then

$$F(x) = f(x) = \| x \|$$

And $\| F \| = \| f \| = 1$

Hence $F(x) = \|x\|$ and $\|F\| = 1$.

2. Let (X, $\| \ \|$) be a normed linear space over a field K (=R or C). Let x_1 and x_2 be two vectors in X such that $x_1 \neq x_2$. Then there is a bounded linear functional F on X such that

$F(x_1) \neq F(x_2)$.

Proof. Given that $x_1 \neq x_2$

$$x = x_1 - x_2 \neq 0, \text{ i.e., } \| x \| = \| x_1 - x_2 \| \neq 0$$

Then there is a bounded linear functional F on X such that

$$F(x) = F(x_1 - x_2) = \| x_1 - x_2 \| \neq 0 \quad (\text{ F is linear})$$

$F(x_1) - F(x_2) \neq 0$

$F(x_1) \neq F(x_2)$.

(3). Let X be a normed linear space over a field K (=R or C) and M be a closed subspace of X. Let $x \in X \setminus M$ then there is a bounded linear functional F on X such that

i. $F(M) = 0 \ \forall m \in M$

ii $F(x) = \text{Dist.} (x, M)$

iii $\| F \| = 1$.

11.5 EXAMPLES

Example 1: Define F on: $R2(R) \to R$ by $F(x, y) = ax + by \ \forall\ x, y \in R^2(R)$

Where a and b are fixed real number

$$\|(x, y)\| = |x| + |y| \quad \forall\ x, y \in R^2(R)$$

$$\|x\| = |x| \ \forall\ x \in R^2(R)$$

Then F is a linear functional. Find $\|F\|$.

Solution. For all $(x_1, y_1), (x_2, y_2) \in R^2(R)$

$$F((x_1, y_1), (x_2, y_2)) = F(x_1 + x_2, y_1 + y_2)$$

$$= a(x_1 + x_2) + b(y_1 + y_2)$$

$$= (ax_1 + by_1) + (ax_2 + by_2)$$

$F((x_1, y_1), (x_2, y_2)) = F(x_1, y_1) + F(x_2, y_2)$

For all $c \in R$, $\forall\ x, y \in R^2(R)$ then

$$F(c(x, y)) = F(cx, cy) = a(cx) + b(cy)$$

$$= c(ax + by)$$

$F(c(x, y)) = c\,F(x, y)$

Hence F is linear functional on $R^2(R)$

$$F(x, y) = ax + by$$

$$|F(x, y)| = |ax + by|$$

$|F(x, y)| \leq |a|\,|x| + |b|\,|y|$

$\leq \max.\{|a|, |b|\}(|x| + |y|)$

$$= \max.\{|a|, |b|\}\,\|(x, y)\|_1$$

Hence F is bounded linear functional on $R^2(R)$

$$F(1, 0) = a,\ |F(1, 0)| = |a| \text{ and } \frac{|F(1,0)|}{|(1,0)|} = |a|$$

Therefore $\|F\| \geq |a|$

F (1, 0) = b, | F(1, 0) | = | b | and $\frac{|F(1,0)|}{|(1,0)|}$ = | b |

Therefore ‖ F ‖ ≥ | b |

Hence ‖ F ‖ ≥ Max. {| a |, |b|}

‖ F ‖ = Max. {| a |, |b|}

11.5.1 PROBLEMS

Problem1: Let $R^2(R)$ be a normed linear space over R and

$\| \ \|_1 : R2(R) \to R$ is defined by

$\| (x, y) \|_1 = | x | + | y | \ \forall\ x, y \in R^2(R)$

Let M = {(x, 0) : x ∈ R }. Then proof that M = { (x, 0) : x ∈ R } is a linear subspace of $R^2(R)$.

Problem 2: If N is a normed linear space and x_0 is a non-empty vector in N, then there exists a functional f_0 in N^* such that

$f_0(x_0) = \| x_0 \|$ and $\| f_0 \| = 1$.

Problem 3: If M is a closed linear subspace of normed linear space N and x_0 is a vector not in M, then there exists a functional f_0 in N^* such that

$F_0(M) = 0$ and $f_0(x_0) \neq 0$.

11.6 SUMMARY

The Hahn-Banach Theorem is a fundamental result in functional analysis, a branch of mathematics. It has several equivalent forms and important implications in various areas of mathematics. Here's a summary of the theorem and its significance.

Hahn Banach theorem for Normed linear space:

Let (X, || ||) be a normed linear space over a filed K (=R or C) and M be a linear subspace of X. Let f be a bounded linear functional on M. Then there is a bounded linear functional F on X such that

(i). $F(x) = f(x) \ \forall x \in M$

(ii). $||F|| = ||f||$

Significance

1. **Extension of Linera Functionals**: The theorem ensures that any bounded linear functional defined on a subspace of a vector space can be extended to the whole space without increasing its norm.
2. **Separation of Convex Sets**: It provides a way to separate disjoint convex sets by a hyperplane, which is crucial in convex analysis and optimization.
3. **Duality Theory**: It forms the basis for duality in optimization problems, particularly in the context of linear programming and convex optimization.
4. **Functional Analysis**: It is a cornerstone in the study of Banach spaces and their duals, leading to the development of various results in functional analysis, such as the existence of continuous linear functionals with specific properties.
5. **Applications**: The theorem has numerous applications in areas such as economics (e.g., utility theory), engineering (e.g., signal processing), and physics (e.g., quantum mechanics), where the extension of functionals and the separation of sets play a key role.

The Hahn-Banach Theorem is celebrated for its generality and thepowerful tools it provides for analysis and problem-solving in mathematics.

11.7 GLOSSARY

i. **Set:** Any well-defined collection of objects or numbers are referred to as a set.

ii. **Interval:** An open interval does not contain its endpoints, and is indicated with parentheses. $(a,b)=]a,b[=\{x\epsilon\mathbb{R}: a<x<b\}$. A closed interval is an interval which contain all its limit points, and is expressed with square brackets. $[a,b]=[a,b]=\{x\epsilon\mathbb{R}: a\le x\le b\}$. A half-open interval includes only one of its endpoints, and is expressed by mixing the notations for open and closed intervals. $(a,b]=]a,b]=\{x\epsilon\mathbb{R}: a<x\le b\}$. $[a,b)=[a,b[=\{x\epsilon\mathbb{R}: a\le x<b\}$.

iii. **Ordered Pairs:** An ordered pair (a, b) is a set of two elements for which the order of the elements is of significance. Thus $(a,b)\neq(b,a)$ unless $a = b$. In this respect (a, b) differs from the set $\{a, b\}$.Again $(a,b)=(c,d)\Leftrightarrow a=c$ and $b=d$ .If X and Y are two sets, then the set of all ordered pairs (x, y), such that $x\in X$ and $y\in Y$ is called Cartesian product of X and Y.

iv. **Relation:** A subset R of $X\times Y$ is called relation of X on Y. It gives a correspondence between the elements of X and Y. If (x, y) be an element of R, then y is called image of x.A relation in which each element of X has a single image is called a function.

v. **Function:** Let X and Y are two sets and suppose that to each element x of X corresponds, by some rule, a single element y of Y. Then the set of all ordered pairs (x, y) is called function.

vi. **Variable:** A symbol such as x or y, used to represent an arbitrary element of a set is called a variable.

vii. **Metric space:** Let $X \neq \emptyset$ be a set then the metric on the set X is defined as a function $d: X \times X \to [0, \infty)$ such that some conditions are satisfied.

viii. **Vector space: -** Let V be a nonempty set with two operations

(i) **Vector addition:** If any $u, v \in V$ then $u + v \in V$

(ii) **Scalar Multiplication:** If any $u \in V$ and $k \in F$ then $ku \in V$

Then V is called a vector space (over the field F) if the following axioms hold for any vectors if the some conditions hold.

CHECK YOUR PROGRESS

Fill in the Blanks:

1. A minimal element of a partially ordered set M is an x ∈ M such that y ~ x impels………….

2. Norm on a vector space X is a ………….. functional on X.

True/False

3. Is the Hahn Banach theorem being true for Complex vector space. True/False.
4. Every semi-normed linear space is normed linear space. True/False.
5. Suppose you have a normed vector space $(X,\|\cdot\|)$and a continuous linear functional defined on a subspace $Y \subseteq X$. Can you extend to the whole space X while preserving its norm? (True / False)
6. Every Vector space have Hamel basis. (True / False)
7. Finite partial order set A has how many maximal elements.
    - **i.** At most one
    - **ii.** Infinite
    - **iii.** Finite
8. A Sublinear functional p satisfies the followings.
    - **i.** P (0) = 0
    - **ii.** $P(-x) \geq - P(x)$.
    - **iii.** Both (i) and (ii).

11.8 REFERENCES

1. E. Kreyszig, (1989), *Introductory Functional Analysis with applications*, John Wiley and Sons.
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3. B. V. Limaye, (2014), *Functional Analysis*, New age International Private Limited.

11.10TERMINAL QUESTIONS

1. If M is a closed linear subspace of normed linear space N and x_0 is a vector not in M, then there exists a functional f_0 in N^* such that $F_0(M) = 0$ and $f_0(x_0) \neq 0$.
2. If N is a normed linear space and x_0 is a non-empty vector in N, then there exists a functional f_0 in N^* such that $f_0(x_0) = \| x_0 \|$ and $\| f_0 \| = 1$.
3. State and proof of Hahn Bacha theorem of real vector space.
4. Sate and proof of Hahn Banach theorem for normed linear space.
5. Let X benormed linear space over a field K (=R or C) and M be a closed subspace of X. Let $x \in X \setminus M$ then there is a bounded linear functional F on X such that
   - **i.** $F(M) = 0 \ \forall m \in M$
   - **ii.** $F(x) = \text{Dist.}(x, M)$
   - **iii.** $\| F \| = 1$.

11.11 ANSWERS

CHECK YOUR PROGRESS

1. y = x.

2. Sub linear functional.

3. True

4. False

5. True

6. True

7. A

8. c.

UNIT 12:

CATEGORY THEOREM

CONTENTS:

12.1. INTRODUCTION

In the beginning of this unit, we will study the Baire’s category theorem. And then we will see that, this theorem led us to three important theorems in functional analysis. More precisely, these theorems are: uniform boundedness theorem, open mapping theorem(in the next unit),closed graph theorem (in the next unit). Hence in this unit we will focus on Baire’s category theorem, uniform boundedness theorem and their applications. It is worth noting that Baire's category theorem has various other applications in functional analysis.

We firstly state the concepts needed for Baire's category theorem. Such concepts are nowhere dense set, first category set and second category set in a metric space. These concept has two names, the students must need to know both names.

12.2 OBJECTIVES

After studying this unit, learner will be able to

- **i.** Understand the Statement and Proof of the Baire’s Category Theorem.
- **ii.** Explained the Statement and Proof of Uniform Boundedness Theorem.
- **iii.** Functional Analysis Applications**:** Apply the theorem in various problems and proofs in functional analysis.
- **iv.** Problem Solving**:** Solve advanced problems in functional analysis and related fields using the Baire’s Category Theorem and Uniform Boundedness Theorem as a tool.

12.3 DEFINITIONS

Definition: A subset $A$of a metric space X is said to be rare (or nowhere dense) in X if its closure $\bar{A}$ has no interior points. i.e. A is said to be rare set if $(\bar{A})^{\circ} = \emptyset$.

Definition: A subset $A$of a metric space X is said to be meager (or of the first category) in X if A is the union of countably many sets each of which is rare in X.

i.e. A is said to be of first category set if $A = \bigcup_{k=1}^{\infty} A_k$, where each A_k are nowhere dense set.

Definition: A subset $A$of a metric space X is said to be non meager (or of the second category) in X if A is not of first category in X. i.e. A is said to be rare set if $(\bar{A})^{\circ} = \emptyset$.

12.4 BAIRE'S CATEGORY THEOREM

Statement: If a metric space X, it is of second category (non meager) in itself. (Hence if X is complete and $X = \bigcup_{k=1}^{\infty} A_k$, where each A_k is closed, then at least one $A_k$has non empty interior.)

Proof. Let X be a complete metric space and$X \neq \emptyset$. On the contrary suppose that X is of first category in itself. Then

$$X = \bigcup_{k=1}^{\infty} A_k$$

with each A_k are nowhere dense set in X. Our aim is to construct a Cauchy sequence (x_k)whose limit x(which exists by completeness of X) is in no A_k, this will give a contradiction.

By assumption, A_1 is nowhere dense in X, so that, by definition, $\bar{A}_1$ does not contain a nonempty open set, but X does (for instance, X itself). Thisimplies that $\overline{A}_1 \neq X$.Hence the complement $\bar{A_1}^c = X - \bar{A}_1$ is not empty and open. We may thus choose a point $x_1$in $\bar{A_1}^c$and an openball about it, say,

$$B_1 = B(x_1;\ \varepsilon_1) \subset \bar{A_1}^c, \varepsilon_1 < \frac{1}{2}.$$

Again, by assumption, A_2 is nowhere dense in X, so that $\bar{A}_2$ does not contain a nonempty open set. Hence it does not contain the open ball $B(x_1; \frac{1}{2}\varepsilon_1)$.This implies that $\overline{A_2}^C \cap B(x_1; \frac{1}{2}\varepsilon_1)$is not empty and open, so that we may choose a point $x_2$and an open ball in this set, say,

$$B_2 = B(x_2;\ \varepsilon_2) \subset \bar{A_2}^c \cap B\left(x_1; \frac{1}{2}\varepsilon_1\right), \varepsilon_2 < \frac{1}{2}\varepsilon_1.$$

By induction we thus obtain a sequence of balls$B_k = B(x_k; \varepsilon_k)\varepsilon_k < 2^{-k}$, such that $B_k \cap A_k = \emptyset$and

$$B_{k+1} \subset B\left(x_k; \frac{1}{2}\varepsilon_k\right) \subset B_k, \text{ for} k = 1,2, \ldots\ldots\ldots\ldots$$

since $\varepsilon_k < 2^{-k}$, the sequence (x_k) is Cauchy sequence and converges, say converges to $x \in X$ because X is complete by assumption. Also, for every m and $n > m$ we have $B_n \subset B\left(x_m; \frac{1}{2}\varepsilon_m\right)$, so that

$$d(x_{m,}x) \leq d(x_m, x_n) + d(x_n, x)$$

$$< \frac{1}{2}\varepsilon_m + d(x_n, x)$$

$$\to \frac{1}{2}\varepsilon_m$$

as $n \to \infty$. Hence $x \in B_m$ for every m. Since $B_m \subset \bar{A}_m{}^c$ we now seethat $x \notin A_m$ for every m, so that $x \notin \cup A_m = X$. This contradicts the fact that $x \in X$.

Remark: Note that there is other form of Baire's theorem which are as follows (without proof):

1. Let $G_1, G_2, \ldots.$ be a sequence of dense open subsets of a complete metric space X. Then $G = \bigcap_{n=1}^{\infty} G_n$ is dense in X.
2. The complement of a meagre subset of a complete metric space is dense. In particular, a complete metric space is of the second category.

Now we are in the state to obtain the uniform boundedness theorem from Baire's category theorem. This theorem states that if X is aBanach space, Y is a normed linear space and a sequence of bounded linear operators $T_n \in B(X, Y)$ is pointwise bounded at every point $x \in X$, then the sequence is uniformly bounded. In other words, pointwise boundedness implies boundedness in some stronger sense, namely, uniform boundedness.

12.5 UNIFORM BOUNDEDNESS THEOREM

Statement: Let (T_n) be a sequence of bounded linear operators $T_n: X \longrightarrow Y$ from a Banach space X into a normed space Y such that (T_n) is pointwise bounded i.e for every $x \in X$, there exists a real number$M_x$such that

$$||T_n x|| \leqq M_x n = 1,2, \ldots \ldots, \qquad \ldots\ldots.(**)$$

where M_x is a real number, the M_x will vary in general with x, and M_x does not depend on n. Then the sequence of the norms $||T_n||$ is bounded, that is, there is a M such that $\left||T_n|\right| \leq M, n = 1,2, \ldots$.

Proof. For every $k \in \mathbb{N}$, let $A_k \subset X$ be the set of all x such that $\left||T_n x|\right| \leq k$, for all n.

A_k is closed. Indeed, for any $x \in \bar{A}_k$ there is a sequence (x_j) in A_k converging to x. This means that for every fixed n we have $\left||T_n x_j|\right| \leq k$ and obtain $\left||T_n x|\right| \leq k$ because T_n is continuous and so is the norm. Hence $x \in A_k$. Thus A_k is closed.

From the equation (**), each $x \in X$ belongs to some A_k Hence

$$X = \bigcup_{k=1}^{\infty} A_{k.}$$

Since X is complete, Baire‘s category theorem implies that some A_k contains an open ball, say,

$$B_0 = B(x_0; r) \subset A_{k_0} \qquad \ldots \ldots \ldots \ldots (** 1)$$

Let $x \in X$ be arbitrary, not zero. We set $z = x_0 + \lambda x,\ \lambda = \frac{r}{2||x||}$. Then $\left||z - x_0|\right| < r$, so that $z \in B_0$. By $(** 1)$ and from the definition of A_{k_0} we thus have $\left||T_n z|\right| \leq k_0$ for all n. Also $\left||T_n x_0|\right| \leq M_0$, since $x_0 \in B_0$. By the definition of z, we get

$$x = \frac{1}{\lambda} (z - xo).$$

This gives for all n

$$||T_n x|| = \frac{1}{\lambda}||T_n(z - x_0|| \leq \lambda(||T_n z|| + ||T_n X_0|| \leq \frac{4}{r}||x||M_0.$$

Hence for all n,

$$||T_n|| = \sup_{||x||=1} ||T_n x|| \leq \frac{4}{r}k_0,$$

Take$M = \frac{4k_0}{r}$, and hence the theorem proved.

12.6 PROBLEMS

Question 1: Let X and Y be normed spaces, with X complete, and let $T_{nm} \in B(X,Y)$,$n,m = 1,2, \dots \dots$ be such that $\overline{\lim_{m\to\infty}}||T_{nm}|| = \infty$, for all $n \in \mathbb{N}$.Then show that there is a set $U \subset X$ of the second category in X such that for $u \in U$, we have $\overline{\lim_{m\to\infty}}||T_{nm}(u)|| = \infty$, for all $n \in \mathbb{N}$

Solution: For a fixed n, let $V_n \subset X$ be the set of vectors v such that $\overline{\lim_{m\to\infty}}||T_{nm}(v)|| < \infty$. Then by the uniform boundedness theorem $V_n \subset$ is of the first category. Therefore $V = \cup_{n=1}^{\infty} V_n$, and thus by remark ** the set $U = X\backslash V$ is of the second category.

Question 2: Show that $\mathbb{N}$ is first category in $\mathbb{R}$ with usual metric, but second category in itself.

Solution: Since in $\mathbb{R}$, every singleton subset is closed and empty interior (in $\mathbb{R}$ every singleton set is an isolated point) and $\mathbb{N} = \cup_{n=1}^{\infty}\{n\}$. Thus $\mathbb{N}$ is first category in $\mathbb{R}$. Now in $\mathbb{N}$ every subset is clopen (closed as well as open set), therefore in $\mathbb{N}$ itself does not have any nowhere dense set. And hence $\mathbb{N}$ in itself can not be written as of union of nowhere dense set. Thus, $\mathbb{N}$ in itself is of second category.

Question 3: Show that any subset of a first category is also first category.

Solution: Let X be a metric space and A be first category in X. Consider $B \subset A$. Since, $A$is fist category in X, therefore $A = \bigcup_{k=1}^{\infty} A_k$, where each $A_k$are nowhere dense sets in X. This implies that closure of$A_k$has empty interior. Also note that interior of closure of $A_k \cap B$contain in $\bar{A_k}^0$, therefore $A_k \cap B$ is also nowhere dense in X with $B = \bigcup_{k=1}^{\infty}(A_k \cap B)$. Hence B is also a first category in X.

Question 4: Countable union of first category sets is again first category.

Solution: Let X be a metric space and for each $n \in \mathbb{N}, A_n$ be first category sets in X. Therefore for each $n \in \mathbb{N}, A_n$can be written as $A_n = \bigcup_{k=1}^{\infty} A_{nk}$, where each $A_{nk}$are nowhere dense sets in X. This implies that $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} A_{nk}$, is again a countable union of nowhere dense set in X. Thus, $\bigcup_{n=1}^{\infty} A_n$ is of first category in X.

Question 5: Give an application of the uniform boundedness theorem.

Solution: The normed space X of all polynomials is not complete, where norm defined by$||x|| = \max_i |a_i|$, $(a_0, a_1, \dots \dots$ the coefficients ofpolynomial x). To prove this we construct a sequence of bounded linear operators on$X$ which are pointwise bounded but not uniformly bounded, so that X cannot be complete.

We may write a polynomial $x \neq 0$ of degree N_x in the form

$$x(t) = \sum_{j=0}^{\infty} a_j t^j$$

$$(a_j = 0 \; for \; j > N_x).$$

As a sequence of operators on X wetake the sequence of functionals $T_n = f_n$ defined by

$$T_n 0 = f_n(0) = 0, \qquad T_n x = f_n(x) = a_0 + a_1 + \cdots + a_{n-1}.$$

By the definition $f_n$is linear. Also,f_n is bounded. Since $|a_j| \le \|x\|$, therefore$|f_n(x)| \leqq n\|x\|$. Furthermore, for each fixed $x \in X$ the sequence $(|f_n x|)$ satisfies the pointwise bounded condition, because a polynomial x of degree N_x, has $N_x + 1$ coefficient, so that by we have,

$$|f_n(x)| = |a_0 + a_1 + \cdots + a_{n-1}|$$

$$\le |a_0| + |a_1| + \cdots + |a_{n-1}|$$

$$\le |a_0| + |a_1| + \cdots + |a_{n-1}| + \cdots + |a_{N_x}|$$

$$\le (N_x + 1)\max_j |a_j| = M_x$$

Hence the sequence $(|f_n x|)$ satisfies the pointwise bounded condition

We now show that (f_n) does not satisfy the uniformly bounded condition, that is, there is no M such that $\|T_n\| = \|f_n\| \le M$ for all n. This we do by choosing particularly polynomialsx_0. For f_n we choose x defined by

$$x(t) = 1 + t + \cdots + t^n.$$

Then $\|x\| = 1$ and

$f_n(x) = 1 + 1 + \cdots + 1 = n = n\|x\|$.

Hence $\|f_n\| \ge \frac{|f_n(x)|}{\|x\|} = n$, so that $(\|f_n\|)$ is unbounded. And hence, the normed space X is not complete.

12.7 SUMMARY

The Baire's Category Theorem and Uniform Boundedness Theorem are a fundamental results in functional analysis, a branch of mathematics. It has several equivalent forms and important implications in various areas of mathematics. The statement of both the theorems are as follows.

Baire's Category Theorem: If a metric space X, it is of second category (non meager) in itself. (Hence if X is complete and $X = \bigcup_{k=1}^{\infty} A_k$, where each A_k is closed, then at least one $A_k$has non empty interior.)

Uniform Boundedness Theorem: Let (T_n) be a sequence of bounded linear operators $T_n: X \longrightarrow Y$ from a Banach space X into a normed space Y such that (T_n) is pointwise bounded i.e for every $x \in X$, there exists a real number$M_x$such that

$$\left||T_n x|\right| \leqq M_x n = 1,2, \dots \dots, \qquad \text{.......(**)}$$

where M_x is a real number, theM_x will vary in general with x, andM_x does not depend on n. Then the sequence of the norms $||T_n||$ is bounded, that is, there is a M such that$\left||T_n|\right| \leq M$,$n = 1,2, \dots$.

12.8 GLOSSARY

1 **Set:** Any well-defined collection of objects or numbers are referred to as a set.

2 **Interval:** An open interval does not contain its endpoints, and is indicated with parentheses. $(a,b) =]a,b[= \{x\epsilon\mathbb{R}: a < x < b\}$. A closed interval is an interval which contain all its limit points, and is expressed with square brackets. $[a,b] = [a,b] = \{x\epsilon\mathbb{R}: a \leq x \leq b\}$. A half-open interval includes only one of its endpoints, and is expressed by mixing the notations for open and closed intervals. $(a,b] =]a,b] = \{x\epsilon\mathbb{R}: a < x \leq b\}$. $[a,b) = [a,b[= \{x\epsilon\mathbb{R}: a \leq x < b\}$.

3 **Ordered Pairs:** An ordered pair (a, b) is a set of two elements for which the order of the elements is of significance. Thus $(a,b) \neq (b,a)$ unless $a = b$. In this respect (a, b) differs from the set $\{a, b\}$.Again $(a,b) = (c,d) \Leftrightarrow a = c$ and $b = d$ .If X and Y are two sets, then the set of all ordered pairs (x, y), such that $x \in X$ and $y \in Y$ is called Cartesian product of X and Y.

4 **Relation:** A subset R of $X \times Y$ is called relation of X on Y. It gives a correspondence between the elements of X and Y. If (x, y) be an element of R, then y is called image of x.A relation in which each element of X has a single image is called a function.

5 **Function:** Let X and Y are two sets and suppose that to each element x of X corresponds, by some rule, a single element y of Y. Then the set of all ordered pairs (x, y) is called function.

6 **Variable:** A symbol such as x or y, used to represent an arbitrary element of a set is called a variable.

7 **Metric space:** Let $X \neq \emptyset$ be a set then the metric on the set X is defined as a function $d: X \times X \rightarrow [0, \infty)$ such that some conditions are satisfied.

8 **Vector space: -** Let V be a nonempty set with two operations

(i) **Vector addition:** If any $u, v \in V$ then $u + v \in V$

(ii) **Scalar Multiplication:** If any $u \in V$ and $k \in F$ then $ku \in V$

Then V is called a vector space (over the field F) if the following axioms hold for any vectors if the some conditions hold.

CHECK YOUR PROGRESS

CYP 1: A subset A is said to be nowhere dense in a metric space X, if …………………

CYP 2: Let Abea second category set in a metric space X. Then : is it true that "$A^c$is of first category"?

CYP 3: True/False:"Ahas an empty interior in a metric space X if and only if $A^c$is dense in X.

CYP4: Write the definition for pointwise bounded for a family of bounded linear operator.

CYP5: Write the definition for uniform bounded for a family of bounded linear operators.

CYP6: True/False: The uniform boundednessof a family of bounded linear operators implies the pointwise boundedness of that family.

CYP7: True/False: Let X & Y be normed spaces and $\{T_1, T_2, \dots, T_n\}$ be finite collection of bounded linear operators from X to Y. Then $\{T_1, T_2, \dots, T_n\}$is pointwise bounded.

CYP8: True/False: Let X & Y be normed spaces and $\{T_1, T_2, \dots, T_n\}$ be finite collection of bounded linear operators from X to Y. Then $\{T_1, T_2, \dots, T_n\}$is uniformly bounded.

CYP9: True/False: Let X & Y be normed spaces and $\{T_1, T_2, \dots, \}$ be countablyinfinite collection of bounded linear operators from X to Y. Then $\{T_1, T_2, \dots, \}$is pointwise bounded.

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3. B. V. Limaye, (2014), *Functional Analysis*, New age International Private Limited.

12.11TERMINAL QUESTIONS

1: If A is nowhere dense in a normed linear space X, and B is non-empty open set in X. Then show that A is nowhere dense in G.

2. Show that superset of a second category set is itself a second category set.

3. State and prove that the Baire's category theorem**.**

4. State and prove that the uniform boundedness theorem**.**

12.12 ANSWERS

CHECK YOUR PROGRESS

1. $(\bar{A})^{\circ} = \emptyset$.
2. Not true, for this take example as $X = \mathbb{R}$**,** $\boldsymbol{A} = [\boldsymbol{0}, \infty)$**.** Then A is of second category, whereas A^c is of second category too.
3. True.
4. Let X & Y be normed spaces and $\{T_\alpha : \alpha \in \Delta\}$ be a family of bounded linear operators $T_\alpha : X \longrightarrow Y$. Then $\{T_\alpha : \alpha \in \Delta\}$is said to be pointwise bounded if for every $x \in X$, there exists a real number$M_x$such that for every $\alpha \in \Delta$

$$||T_\alpha x|| \leqq M_x.$$

5. Let X & Y be normed spaces and $\{T_\alpha : \alpha \in \Delta\}$ be a family of bounded linear operators $T_\alpha : X \longrightarrow Y$. Then $\{T_\alpha : \alpha \in \Delta\}$is said to be uniformbounded ifthere exists a real number$M$such that for every $\alpha \in \Delta$

$$||T_\alpha|| \leqq M.$$

6. True.
7. True.
8. True.
9. May not be true.

UNIT 13:

OPEN MAPPING THEOREM

AND CLOSED GRAPH THEOREM

CONTENTS:

13.1 INTRODUCTION

In previous units Hahn-Banach theorem and Category theorem defined in a simple manner. Now, in this unit Open mapping theorem and Closed Graph Theorem defined in a systematic manner.

In functional analysis, the open mapping theorem, also known as the Banach – Schauder theorem or the Banach theorem (named after Stefan Banach and Juliusz Schauder), is a fundamental result that states that if a bounded or continuous linear operator between Banach spaces is surjective then it is an open map

In functional analysis and topology, the closed graph theorem is a output of connecting the continuity of certain kinds of functions to a topological property of their curve. Mainly, the theorem gives a linear operator between two Banach spaces is continuous if and only if the graph of the operator is closed (such an operator is called a closed linear operator; see also closed graph property).

The closed graph theorem has important application throughout functional analysis, because it can control whether a partially-defined linear operator admits continuous extensions. For this cause, it has been generalized to many circumstances beyond the elementary formulation above.

We are assuming that the learners are familiar with different concept of analysis such as closures, interiors of set, dense set, separable metric space, no-where dense set. These concepts are defined in advanced real analysis in first semester.

13.2 OBJECTIVES

After studying this unit, learner will be able to

i. Describe the statement of open mapping theorem and closed graph theorem.
ii. Explain the proof of open mapping theorem and closed graph theorem.
iii. Understand the concept of open mapping and closed linear operator.

13.3 OPEN MAPPING

Let X and Y be metric spaces. Then $T: D(T) \longrightarrow Y$ with domain $D(T) \subset X$ is called an open mapping if for every open set in $D(T)$ the image is an open set in Y.

Note that if a mapping is not surjective, one must take care to distinguish between the assertions that the mapping is open as a mapping from its domain

a) into Y,

b) onto its range.

b) is weaker than a). For instance, if $X \subset Y$, the mapping $x \mapsto x$ of X into Y is open if and only if X is an open subset of Y, whereas the mapping $x \mapsto x$ of X onto its range (which is X) is open in any case.

13.4 OPEN MAPPING THEOREM, BOUNDED INVERSE THEOREM

We have discussed the Hahn-Banach theorem and the uniform boundedness theorem and shall now approach the third "big" theorem in this unit, the pen mapping theorem. It will be concerned with open mappings. These are mappings such that the image of every open set is an open set. Remembering our discussion of the importance of open sets, we understand that open mappings are of general interest. More specifically, the open mapping theorem states conditions under which a bounded linear operator is an open mapping. As in the uniform boundedness theorem we again need completeness, and the present theorem exhibits another reason why Banach spaces are more satisfactory than incomplete normed spaces. The theorem also gives conditions under which the inverse of a hounded linear operator is bounded. The proof of the open mapping theorem will be based on Baire's category theorem stated and explained in previous unit.

13.4.1 STATEMENT

A bounded linear operator T from a Banach space X onto a Banach space Y is an open mapping. Hence if T is bijective, T^{-1} is continuous and thus bounded.

Or

in other words every bounded linear transformation from a Banach space onto a Banach space is open.

13.4.2 LEMMA (OPEN UNIT BALL)

A bounded linear operator T from a Banach space X onto a Banach space Y has the property that the image $T(B_0)$ of the open unit ball

$$B_0 = B(0; 1) \subset X$$

contains an open ball about $0 \in Y$.

Proof. We prove the lemma by following steps:

a) The closure of the image of the open ball $B_0 = B\left(0; \frac{1}{2}\right)$ contains an open ball B^*.

b) $\overline{T(B_n)}$ contains an open ball V_n about $0 \in Y$, where

$$B_n = B(0;\ 2^{-n}) \subset X.$$

c) $T(B_0)$ contains an open ball about $0 \in Y$.

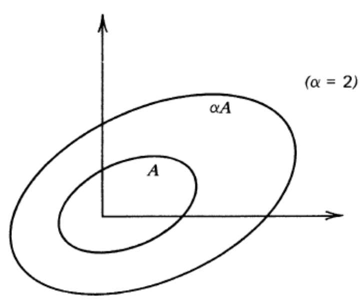

Illustration of formula (1)

Fig.13.4.2.1

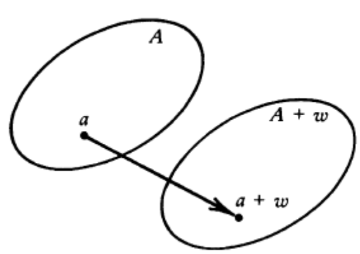

Illustration of formula (2)

Fig.13.4.2.2

a) In connection with subsets $A \subset X$ we shall write αA (α a scalar) and $A + w$ ($w \in X$) to mean

$$(1) \qquad \alpha A = \{x \in X \mid x = \alpha a, a \in A\}$$

$$(2) \qquad A + w = \{x \in X \mid x = a + w, a \in A\}$$

and similarly for subsets of Y.

We consider the open ball $B_0 = B(0;1) \subset X$. Any fixed $x \in X$ is in kB_1 with real k sufficiently large $(k > 2\|x\|)$.

Hence,

$$X = \bigcup_{k=1}^{\infty} kB_1$$

Since T is surjective and linear.

$$(3) \qquad Y = T(X) = T\left(\bigcup_{k=1}^{\infty} kB_1\right) = \bigcup_{k=1}^{\infty} kT(B_1) = \bigcup_{k=1}^{\infty} \overline{kT(B_1)}.$$

Note that by taking closures we did not add further points to the union since that union was already the whole space$Y$. Since Y is complete, it is non meager in itself, by Baire's category theorem.

Hence $\overline{kT(B_1)}$ must contain some open ball. This implies that $\overline{T(B_1)}$ also contains an open ball, $B^* = B(y_0;\varepsilon) \subset \overline{T(B_1)}$. It follows that,

$$(4) \qquad B^* - y_0 = B(0;\varepsilon) \subset \overline{T(B_1)} - y_0.$$

(b) We prove that $B^* - y_0 \subset \overline{T(B_0)}$, where Bo is given in the theorem. This we do by showing that

(5) $$\overline{T(B_1)} - y_0 \subset \overline{T(B_0)}.$$

Let $y \in \overline{T(B_1)} - y_0$.

Then $y + y_0 \in \overline{T(B_1)}$, and we remember that, $y_0 \in \overline{T(B_1)}$, there are

$$u_n = Tw_n \in T(B_1)$$

such that

$$u_n \longrightarrow y + y_0,$$

$$v_n = Tz_n \in T(B_1)$$

such that

$$v_n \longrightarrow y_0$$

Since $w_n, z_n \in B_1$ and B_1 has radius $\frac{1}{2}$, it follows that

$$\|w_n - z_n\| \leqq \|w_n\| + \|z_n\| < \tfrac{1}{2} + \tfrac{1}{2} = 1,$$

So that,

$$w_n - z_n \in B_0.$$

From

$$T(w_n - z_n) = Tw_n - Tz_n = u_n - v_n \longrightarrow y$$

We see that, $y \in \overline{T(B_0)}$.

Since $y \in \overline{T(B_1)} - y_0$ was arbitrary, this proves (5). From (4) we thus have,

(6) $$B^* - y_0 = B(0; \varepsilon) \subset \overline{T(B_0)}.$$

Let

$$B_n = B(0; 2^{-n}) \subset X.$$

Since T is linear,

$$\overline{T(B_n)} = 2^{-n}\, \overline{T(B_0)}$$

From (6) we thus obtain (7),

$$(7) \qquad V_n = B(0;\ \varepsilon/2^n) \subset \overline{T(B_n)}.$$

(c) We finally prove that

$$V_1 = B(0; \tfrac{1}{2}\varepsilon) \subset T(B_0)$$

By showing that every $y \in V_1$ is in $T(B_0)$. So let $y \in V_1$. From (7) with n = 1 we have, $V_1 \subset \overline{T(B_1)}$.

Hence $y \in \overline{T(B_1)}$.

There must be a

$$v \in T(B_1)$$

close to y, say,

$$\|y - v\| < \varepsilon/4.$$

Now

$$v \in T(B_1)$$

implies $v = Tx_1$ for some $x_1 \in B_1$, hence,

$$\|y - Tx_1\| < \frac{\varepsilon}{4}.$$

From this and (7) with n = 2 we see that $-Tx_1 \in V_2 \subset \overline{T(B_2)}$. As before we conclude that there is an $x_2 \in B_2$ such that

$$(8) \qquad \left\| y - \sum_{k=1}^{n} Tx_k \right\| < \frac{\varepsilon}{2^{n+1}} \qquad (n = 1, 2, \cdots).$$

Let

$$z_n = x_1 + \cdots + x_n.$$

Since

$$x_k \in B_k,$$

we have

$$\|x_k\| < 1/2^k.$$

This yields for $n > m$,

$$\|z_n - z_m\| \leqq \sum_{k=m+1}^{n} \|x_k\| < \sum_{k=m+1}^{\infty} \frac{1}{2^k} \longrightarrow 0$$

as

$$m \longrightarrow \infty.$$

Hence (z_n) is Cauchy. (z_n) converges, say,

$$z_n \longrightarrow x$$

because X is complete.

Also B_0 has radius 1 and,

$$\sum_{k=1}^{\infty} \|x_k\| < \sum_{k=1}^{\infty} \frac{1}{2^k} = 1. \tag{9}$$

Since T is continuous,

$$Tz_n \longrightarrow Tx,$$

and (8) shows that

$$Tx = y.$$

Hence,

$$y \in T(B_0).$$

13.4.3 PROOF OF THE THEOREM

We are using above lemma for the proof of the theorem.

Statement:

A bounded linear operator T from a Banach space X onto a Banach space Y is an open mapping. Hence if T is bijective, T^{-1} is continuous and thus bounded.

Proof. We prove that for every open set $A \subset X$ the image $T(A)$ is open in Y.
This we do by showing that for every $y = Tx \in T(A)$ the set $T(A)$ contains an open ball about $y = Tx$.
Let

$$y = Tx \in T(A).$$

Since A is open, it contains an open ball with center x.
Hence $A - x$ contains an open ball with center 0; let the radius of the ball be r and set $k = \frac{1}{r}$, so that $r = \frac{1}{k}$.
Then $k(A - x)$ contains the open unit ball $B(0; 1)$.
By previous lemma implies that $T(k(A - x)) = k[T(A) - TX]$ contains an open ball about 0, and so does $T(A) - Tx$.
Hence $T(A)$contains an open ball about $Tx = y$.
Since $y \in T(A)$ was arbitrary, $T(A)$ is open.
Finally, if $T^{-1}: Y \to X$ exists, it is continuous because T is open. Since $T^{-1}$is linear by it is bounded . (We have read this theorem in previous studies).

13.5 CLOSED LINEAR OPERATOR

Let X and Y be normed spaces and $T: D(T) \longrightarrow Y$ is a linear operator with domain $D(T) \subset X$. Then T is called a closed linear operator if its graph

$$\mathcal{G}(T) = \{(x, y) \mid x \in \mathfrak{D}(T),\ y = Tx\}$$

is closed in the normed space $X \times Y$, where the two algebraic operations of a vector space in $X \times Y$ are defined as usual, that is

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

$$\alpha(x, y) = (\alpha x, \alpha y)$$

(a a scalar) and the norm on $X \times Y$ is defined by,

$$(1) \qquad \|(x, y)\| = \|x\| + \|y\|.$$

13.6 CLOSED GRAPH THEOREM

Not all linear operators of practical importance are bounded. For instance, the differential operator is unbounded, and in quantum mechanics and other applications one needs unbounded operators quite frequently. However, practically all of the linear operators which the analyst is likely to use are so-called closed linear operators. This makes it worthwhile to give an introduction to these operators. In this unit we define closed linear operators on normed spaces and consider some of their properties, in particular in connection with the important closed graph theorem which states sufficient conditions under which a closed linear operator on a Banach space is bounded.

Statement: Let X and Y be Banach spaces and $T: D(T) \longrightarrow Y$ a closed linear operator, where $D(T) \subset X$. Then if $D(T)$ is closed in X, the operator T is bounded.

Proof. We first show that $X \times Y$ with norm defined by (1) is complete.
Let (z_n) be Cauchy in $X \times Y$, where $z_n = (x_n, y_n)$. Then for every $\varepsilon > 0$ there is an N such that,

$$(2) \qquad \|z_n - z_m\| = \|x_n - x_m\| + \|y_n - y_m\| < \varepsilon \qquad (m, n > N).$$

Hence (x_n) and (y_n) are Cauchy in X and Y, respectively, and converge.

$$x_n \longrightarrow x$$

and

$$y_n \longrightarrow y,$$

because X and Y are complete.

This implies that,

$$z_n \longrightarrow z = (x, y)$$

since from (2) with $m \to \infty$ we have,

$$\|z_n - z\| \leqq \varepsilon$$

for

$$n > N.$$

Since the Cauchy sequence (z_n) was arbitrary, $X \times Y$ is complete.

By assumption, $\mathscr{G}(T)$ is closed in $X \times Y$ and $\mathscr{D}(T)$ is closed in X.

Hence $\mathscr{G}(T)$ and $\mathscr{D}(T)$ are complete

We now consider the mapping:

$$P\colon\ \mathscr{G}(T) \longrightarrow \mathscr{D}(T)$$

$$(x, Tx) \longmapsto x.$$

P is linear. P is bounded because

$$\|P(x, Tx)\| = \|x\| \leqq \|x\| + \|Tx\| = \|(x, Tx)\|.$$

P is bijective; in fact the inverse mapping is

$$P^{-1}\colon\ \mathscr{D}(T) \longrightarrow \mathscr{G}(T)$$

$$x \longmapsto (x, Tx).$$

Since $\mathscr{G}(T)$ and $\mathscr{D}(T)$ are complete,

we can apply the bounded inverse theorem,

and see that

P^{-1} is bounded, say, $\|(x, Tx)\| \leqq b\|x\|$ for and see that P^{-1} is bounded, say, $\|(x, Tx)\| \leqq b\|x\|$ for some b and all $x \in \mathfrak{D}(T)$. Hence T is bounded because

$$\|Tx\| \leqq \|Tx\| + \|x\| = \|(x, Tx)\| \leqq b\|x\|$$

For all $x \in D(T)$.

By definition, $\mathcal{G}(T)$ is closed if and only if $z = (x, y) \in \overline{\mathcal{G}(T)}$ implies $z \in \mathcal{G}(T)$.

we see that $z \in \overline{\mathcal{G}(T)}$ if and only if

there are $z_n = (x_n, Tx_n) \in \mathcal{G}(T)$ such that $z_n \longrightarrow z$, hence

(3) $$x_n \longrightarrow x, \qquad Tx_n \longrightarrow y;$$

and $z = (x, y) \in \mathcal{G}(T)$ if and only if $x \in \mathfrak{D}(T)$ and $y = Tx$.

This proves the following useful criterion which expresses a property that is often taken as a definition of closedness of a linear operator.

13.6 .1 THEOREM (CLOSED LINEAR OPERATOR)

Let $T: D(T) \longrightarrow Y$ be a linear operator, where $D(T) \subset X$ and X and Y are normed spaces. Then T is closed if and only if it has the following property.

If $x_n \to x$, where $x_n \in D(T)$ and $Tx_n \to y$ then $x \in D(T)$ and $Tx = y$.

13.6 .2 EXAMPLE (DIFFERENTIAL OPERATOR)

Let $X = C[0, 1]$ and

$$T: \mathfrak{D}(T) \longrightarrow X$$

$$x \longmapsto x'$$

where the prime denotes differentiation and $\mathfrak{D}(T)$ is the subspace of functions $x \in X$ which have a continuous derivative. Then T is not bounded, but is closed.

We are showing that *T* is not bounded:

Let X be the normed space of all polynomials on $J = [0, 1]$ with norm given $\|x\| = \max |x(t)|$, $t \in J$. A differentiation operator T is defined on X by

$$Tx(t) = x'(t)$$

where the prime denotes differentiation with respect to t. This operator is linear but not bounded. Indeed, let $x_n(t) = t^n$, where $n \in \mathbf{N}$. Then $\|x_n\| = 1$ and

$$Tx_n(t) = x_n'(t) = nt^{n-1}$$

so that $\|Tx_n\| = n$ and $\|Tx_n\|/\|x_n\| = n$. Since $n \in \mathbf{N}$ is arbitrary, this shows that there is no fixed number c such that $\|Tx_n\|/\|x_n\| \leqq c$. From this and

(1) we conclude that T is not bounded.

Let (x_n) in $\mathfrak{D}(T)$ be such that

both (x_n) and (Tx_n) converge, say,

$$x_n \longrightarrow x \qquad \text{and} \qquad Tx_n = x_n' \longrightarrow y.$$

Since convergence in the norm of $C[0, 1]$ is uniform convergence on $[0, 1]$, from $x_n' \longrightarrow y$ we have

$$\int_0^t y(\tau)\, d\tau = \int_0^t \lim_{n\to\infty} x_n'(\tau)\, d\tau = \lim_{n\to\infty} \int_0^t x_n'(\tau)\, d\tau = x(t) - x(0),$$

that is,

$$x(t) = x(0) + \int_0^t y(\tau)\, d\tau.$$

This shows that $x \in \mathfrak{D}(T)$ and $x' = y$.

Using the theorem 13.6 it is implies that T is closed.

It is worth noting that in this example, $\mathfrak{D}(T)$ is not closed in X since T would then be bounded by the closed graph theorem.

Remark: Closedness does not imply boundedness of a linear operator.

Conversely, boundedness does not imply closedness.

Proof. The first statement is illustrated by 13.6 and the second one by the following example.

We are taking

$$T:\ \mathfrak{D}(T) \longrightarrow \mathfrak{D}(T) \subset X$$

identity operator on $\mathfrak{D}(T)$, where $\mathfrak{D}(T)$ is a proper dense subspace of a normed space X. Then it is trivial that T is linear and bounded. However, T is not closed. This follows immediately from Theorem if we take an $x \in X - \mathfrak{D}(T)$ and a sequence (x_n) in $\mathfrak{D}(T)$ which converges to x.

13.6.3 LEMMA(CLOSED OPERATOR)

Let T: $\mathfrak{D}(T) \longrightarrow Y$ be a bounded linear operator with domain $\mathfrak{D}(T) \subset X$, where X and Y are normed spaces. Then:

(a) *If $\mathfrak{D}(T)$ is a closed subset of X, then T is closed.*

(b) *If T is closed and Y is complete, then $\mathfrak{D}(T)$ is a closed subset of X.*

Proof. **(a)** If (x_n) is in $\mathfrak{D}(T)$ and converges, say, $x_n \longrightarrow x$, and is such that (Tx_n) also converges, then $x \in \overline{\mathfrak{D}(T)} = \mathfrak{D}(T)$ since $\mathfrak{D}(T)$ is closed, and $Tx_n \longrightarrow Tx$ since T is continuous. Hence T is closed

(b) For $x \in \overline{\mathfrak{D}(T)}$ there is a sequence (x_n) in $\mathfrak{D}(T)$ such that $x_n \longrightarrow x$;

Since T is bounded,

$$\|Tx_n - Tx_m\| = \|T(x_n - x_m)\| \leqq \|T\| \, \|x_n - x_m\|.$$

This shows that (Tx_n) is Cauchy. (Tx_n) converges, say, $Tx_n \longrightarrow y \in Y$

Since T is closed, $x \in \mathfrak{D}(T)$.

[Using the theorem 13.6.1]

Hence $\mathfrak{D}(T)$ is closed because $x \in \overline{\mathfrak{D}(T)}$ was arbitrary.

13.7 SUMMARY

In this unit we are explaining Open mapping, Open mapping theorem, Bounded Inverse Theorem Statement, Lemma related to Open unit ball and then the gives Proof of the open mapping theorem. After that Closed linear operator defined then Closed Graph Theorem state and prove. In continuation theorem related to Closed linear operator gives then Example(Differential operator) defined. After that Lemma related to Closed operator defined in a proper manner.

13.8 GLOSSARY

i. **Set:** Any well-defined collection of objects or numbers are referred to as a set.

ii. **Interval:** An open interval does not contain its endpoints, and is indicated with parentheses. $(a,b)=]a,b[=\{x\epsilon\mathbb{R}: a<x<b\}$. A closed interval is an interval which contain all its limit points, and is expressed with square brackets. $[a,b]=[a,b]=\{x\epsilon\mathbb{R}: a\leq x\leq b\}$. A half-open interval includes only one of its endpoints, and is expressed by mixing the notations for open and closed intervals. $(a,b]=]a,b]=\{x\epsilon\mathbb{R}: a<x\leq b\}$. $[a,b)=[a,b[=\{x\epsilon\mathbb{R}: a\leq x<b\}$.

iii. **Ordered Pairs:** An ordered pair (a, b) is a set of two elements for which the order of the elements is of significance. Thus $(a,b)\neq(b,a)$ unless a = b. In this respect (a, b) differs from the set $\{a,\ b\}$.Again $(a,b)=(c,d)\Leftrightarrow a=c$ and $b=d$ .If X and Y are two sets, then the set of all ordered pairs (x, y), such that $x\in X$ and $y\in Y$ is called Cartesian product of X and Y.

iv. **Relation:** A subset R of $X\times Y$ is called relation of X on Y. It gives a correspondence between the elements of X and Y. If (x, y) be an element of R, then y is called image of x.A relation in which each element of X has a single image is called a function.

v. **Function:** Let X and Y are two sets and suppose that to each element x of X corresponds, by some rule, a single element y of Y. Then the set of all ordered pairs (x, y) is called function.

vi. **Variable:** A symbol such as x or y, used to represent an arbitrary element of a set is called a variable.

vii. **Metric space:** Let $X \neq \emptyset$ be a set then the metric on the set X is defined as a function $d: X \times X \to [0, \infty)$ such that some conditions are satisfied.

viii. **Vector space: -** Let V be a nonempty set with two operations

(i) **Vector addition:** If any $u, v \in V$ then $u + v \in V$

(ii) **Scalar Multiplication:** If any $u \in V$ and $k \in F$ then $ku \in V$

Then V is called a vector space (over the field F) if the following axioms hold for any vectors if the some conditions hold.

Ix Normed Space

X Banach Space

XI Linear operator

XII Linear functional

CHECK YOUR PROGRESS

FILL IN THE BLANKS

1. A bounded linear operator T from a Banach space X onto a Banach space Y is an open mapping. Hence if T is………, T^{-1} is continuous and thus bounded.

2. Let X and Y be Banach spaces and $T: D(T) \longrightarrow Y$ a closed linear operator, where $D(T) \subset X$. Then if $D(T)$ is …….. in X, the operator T is bounded.

CHOOSE THE CORRECT ONE

3.

A bijective map $A : X \to Y$ is open if and only if :

(a) $A : X \to Y$ is invertible.

(b) $A : X \to Y$ is bounded.

(c) $A^{-1} : Y \to X$ is bounded.

(d) $A^{-1} : Y \to X$ is open.

4.

Every complete subspace of a normed space is:

(a) closed.

(b) open

(c) finite

(d) None of these.

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13.9 TERMINAL QUESTIONS

1. State and prove open mapping theorem.

………………………………………………………………

…………………………………………………………………

2. State and prove closed graph theorem

……………………………………………………………………..

………………………………………………………………………

………………………………………………………………………

13. 10 ANSWERS

CHECK YOUR PROGRESS

1. Bijective
2. Closed
3. c
4. **a**

UNIT 14:

BANACH FIXED POINT THEOREM

CONTENTS

14.1 INTRODUCTION

Fixed point theory is an important branch of Mathematics. The presence or absence of a fixed point is an intrinsic property of a map. However, many necessary or sufficient conditions for existence of such points involve a mixture of algebraic, order theoretic or topological properties of the mappings or its domain.

14.2 OBJECTIVES

After completion of this unit, learner will be able to

1. Analyze about fixed point.
2. Describe the contraction mapping.
3. Understand the existence of fixed point.
4. Prove some important fixed point theorems.

14.3 FIXED POINT

Definition.

Let X be a non empty set and $T : X \to X$ be a map. A point $x_0 \in X$ is called a fixed point of T if $Tx_0 = x_0$.

Examples:

- Let α be any non zero real number and Tα : R → R be defined as

$$T_\alpha(x) = x + \alpha.$$

Then T_α has no fixed point in R.

- Let T : R → R be defined as T(x) = x^2. Then 0 and 1 are two fixed points of T.
- Let T : $R^2 \to R^2$ be defined as T(x,y) = (x,0). Then T has infinitely many fixed points (all points of the x-axis).

14.4 CONTRACTION AND OTHER MAPPINGS

- **Lipschitzian mapping:** A mapping f on a metric space (X, d),$\forall\, x, y \in X$ is a Lipschitzian mapping if there exists a real number $\alpha > 0$ such that

$d(Tx, Ty) \leq d(x, y)$…….(a)

- **Contraction mapping:** A mapping$f$ on a metric space (X, d), $\forall\, x, y \in X$ is a Contraction Mapping if there exists a real numberα, $0 \leq \alpha < 1$, such that

$d(Tx, Ty) \leq \alpha\, d(x, y)$ ....(1)

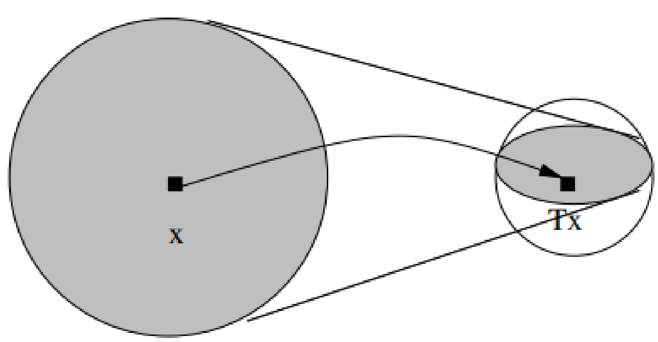

Fig 14.4.1

- **Non-expensive mapping:** A mapping f on a metric space $(X, d), \forall\, x, y \in X$ is a Non-expensive mapping if
  $d\,(Tx, Ty) \leq d(x, y)$ ……..(b)

- **Contractive Mapping:** A mapping f on a metric space $(X, d), \forall\, x, y \in X$ is a contractive mapping if
  $d\,(Tx, Ty) < d(x, y)$ ………..(c)

It is important to note that:

$$Contraction \Rightarrow non-expansive \Rightarrow Lipschitz \Rightarrow Contractive,$$

While the opposite of what it implies is untrue.

Example:

- The identity mapping $I: X \to X$, is non-expansive but not contractive as $\forall\, x, y \in X$,

$$d(Ix, Iy) \leq d(x, y).$$

- Mapping $f: X \to X$ defined by

$$f(x) = x + \frac{1}{x}, \forall\, x \in X$$

  Is a contractive mapping while f is not a contraction.

- Mapping $f: X \to X$ defined by
  $f(x) = 3x,$

  T is a Lipschitzian mapping for $M = 3$, while f is not a contraction.

CHECK YOUR PROGRESS

1. Describe fixed point...............................
2. T_1 : R → R be defined as $T_1(x) = x + 1$. Then T_1 has .... fixed point in R.

14.5 BANACH FIXED POINT THEOREM (CONTRACTION)

Theorem 1. Consider a metric space $X = (X, d)$, where $X \neq \emptyset$. Suppose that X is a complete and let $T: X \longrightarrow X$ be a contraction on X. Then T has precisely one (unique) fixed point.

Proof. Construct a sequence (x_n) and show that it is Cauchy, so that it converges in the complete space X, and then we prove that its $limit\ x$ is a fixed point of T and T has no further fixed points. This is the explanation of the proof.

We choose any $x_n \in X$ and define the "iterative sequence" (x_n) by

$$x_0,\ x_1 = Tx_0,\ x_2 = Tx_1 = T^2x_0, \dots \dots x_n = T^n x_0 \dots \dots \quad \dots\dots(2)$$

Clearly, this is the sequence of the image of x_0 under repeated application of T.

We show that (x_n) is Cauchy.

From equation (1) and equation (2),

$$d(Tx, Ty) = d(Tx_m, Tx_{m-1})$$
$$\leq \alpha\, d(x_m, x_{m-1})$$

$$= \alpha\, d(Tx_{m-1}, Tx_{m-2})$$
$$\leqq \alpha^2\, d(x_{m-1}, x_{m-2})$$
$$\ldots\ldots\ldots\ldots\ldots\leqq \alpha^m\, d(x_1, x_0)\ldots\ldots\ldots\ldots\ldots(3)$$

Hence by triangle inequality and the formula for the sum of a geometric progression we obtain for $n > m$.

$$d(x_m, x_n) \leqq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots d(x_{n-1}, x_n)$$
$$\leqq (\alpha^m + \alpha^{m+1} + \cdots + \alpha^{n-1}) d(x_0, x_1) =$$
$$\alpha^m \frac{1-\alpha^{m-n}}{1-\alpha} d(x_0, x_1).$$

Since $0 < \alpha < 1$, in the numerator we have $1 - \alpha^{n-m} < 1$. Consequently,

$$d(x_m, x_n) \leqq \frac{\alpha^m}{1-\alpha} d(x_0,x_1) \qquad (n > m).$$

On the right, $0 < \alpha < 1$ and $d(x_0, x_1)$ is fixed, so that we can make the right-hand side as small as we please by taking m sufficiently large (and $n > m$). This proves that (x_m) is Cauchy. Since X is complete, (x_m) converges, say $x_m \to x$. We show that this limit x is a fixed point of the mapping T.

From the triangle inequality and (1) we have,

$$d(x, Tx) \leqq d(x, x_m) + d(x_{m+1}, Tx).$$
$$\leqq d(x, x_m) + \alpha d(x_{m-1}, x),$$

and can make the sum in the second line smaller than any preassigned $\in > 0$ because $x_m \to x$. We conclude that $d(x, Tx) = 0$, so that $x = Tx$ (By second property of metric space). This shows that x is a fixed point of the mapping T.

x is the only fixed point of the mapping T because $Tx = x$ and $T\bar{x} = \bar{x}$ we obtain by (1),

$$d(x, \bar{x}) = d(Tx, T\bar{x}) \leqq \alpha d(x, \bar{x}),$$

which implies $d(x, \bar{x}) = 0$ since $\alpha < 1$.

Hence $x = \bar{x}$ (By second property of metric space).

Then it means T has precisely one (unique) fixed point.

CHECK YOUR PROGRESS

3. Converse of Banach fixed point theorem is always true

True/False

14.6.1 COROLLARY (ITERATION, ERROR BOUNDS)

Under the condition of previous theorem (Banach contraction principle) the iterative sequence (2) with arbitrary $x_0 \in X$ converges to the unique fixed point x of T. Error estimates are the prior estimate

$$d(x_m, x) \le \frac{\alpha^m}{1-\alpha} d(x_0, x_1)$$

…………………………………….(5)

and the posterior estimate

$$d(x_m, x) \le \frac{\alpha}{1-\alpha} d(x_{m-1}, x_m)$$

…………………………………….(6)

Proof. The first statement is obvious from the previous proof. Inequality (5) follows from (4) by letting $n \to \infty$.

We derive (6). Taking $m = 1$ and the writing y_0 for x_0 and y_1 for x_1, we have from (5),

$$d(y_1, x) \le \frac{\alpha}{1-\alpha} d(y_0, y_1).$$

Setting $y_0 = x_{m-1,}$ we have $y_1 = Ty_0 = x_m$ and obtain (6).

14.6.2 IMPORTANT THEOREMS

Theorem 2. Let T be a mapping of a complete metric space $X = (X, d)$ into itself. Suppose T is a contraction on a closed ball $Y = \{x | d(x, x_0) \leqq r\}$,
that is, T satisfies (1) for all $x, y \in Y$. Moreover, assume that

$$d(x, Tx_0) < (1 - \alpha)r.$$

……………………………………………………………………….(7)

Then the iterative sequence (2) converges to an $x \in Y$. This x is a fixed point of T and is the only fixed point of T in Y.

Proof . We merely have to show that all $x_m{}'s$ as well as x lie in Y. We put $m = 0$ in (4), change n to m and use (7) to get

$$d(x_0, x_m) \leq \frac{1}{1 - \alpha} d(x_0, x_1) < r.$$

Hence all $x_m{}'s$ are in Y. Also $x \in Y$ since
$x_m \longrightarrow x$ and Y is closed. The assertion of the theorem now follows from the proof of Banach Theorem.

Theorem 3. Every contraction mapping is continuous

Proof.

Let $T : X \to X$ be a contraction on a metric space (X, d),

with modulus β, and let

$\overline{x} \in X$. Let $\epsilon > 0$, and let $\delta = \epsilon$.

and let $\delta = \epsilon$. Then $d(x, \overline{x}) < \delta \Rightarrow d(Tx, T\overline{x}) \leqq \beta\delta < \epsilon$. Therefore T is continuous at $\overline{x}$. Since $\overline{x}$ was arbitrary, T is continuous on X.

Theorem 4. Every contraction mapping is uniformly continuous.

Theorem 5.Brouwer fixed Point Theorem. Let f: S $\rightarrow$S be a continuous function from a non – empty, compact, convex set $S \subset R^n$ into itself, then there exists a $X \in S$, such that $X = f(X)(X$ is a fixed of function f).

Theorem 6. Schauder fixed point theorem. Let S be a non empty closed convex subset of a normed space X. Then every continuous function from S into a compact subset of S has a fixed point.

Theorem 7. Markov-Kakutani theorem. Let C be a non empty compact convex subset of a normed linear space X; and T a family of a affine continuous maps from C to C such that FG = GF for all F,G $\in$T. Then the family T has a common fixed point in C.

Theorem 8. Browder fixed point theorem. [11] . Let X be a uniformly convex Banach space and S be a non empty closed bounded and convex subset of X. If T is a family of non expansive maps from S to S such that FG = GF for all F; G $\in$T; then T has a common fixed point in S.

Theorem 9. Let T be a continuous mapping of a complete metric space $X,$ into itself such that T^k is a contraction mapping of X for some positive integer k. Then T has a unique fixed point.

Proof. Since from theorem 1 we can say that T^k has a unique fixed point u in X and,

$u = \lim_{n\to\infty} (T^k)^n x_0 \in X$.

Also $\lim_{n\to\infty} (T^k)^n (Fx_0) = u$. Hence,

$$u = \lim_{n\to\infty}(T^k)^n(Fx_0) = \lim_{n\to\infty} T(T^k)^n x_0 = T\left(\lim_{n\to\infty}(T^k)^n x_0\right).$$

Since each fixed point of T is also a fixed point of F^k. The uniqueness of the fixed point of T follows from the uniqueness of the fixed point of F^k.

Remark : The continuity condition on T is not necessary.

Let $X = \mathbb{R}, T(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational.} \end{cases}$

T is not continuous mapping and hence not a contraction mapping.

But $T^2(x) = \begin{cases} f(1) = 1, & x \text{ is rational} \\ f(0) = 1, & x \text{ is irrational.} \end{cases}$

T^2 is a contraction and T^2 and T both have the same fixed point 1.

Theorem 10. Suppose (X, d) is a complete metric space and suppose $T: X \longrightarrow X$ is a mapping for which T^N is a contraction mapping of X for some positive integer N. Then T has a unique fixed point.

Proof. By Banach contraction theorem T^N has a unique fixed point x.

However,

$$T^{N+1}(x) = T\big(T^N(x)\big) = T(x),$$

so $T(x)$ is a also a fixed point of T^N.

Since the fixed point of T^N is unique, it must be the case that $T(x) = x$.

Also, if $T(y) = y$ then $T^N(y) = y$ proving (again by uniqueness) that $y = x$.

14.6 EXAMPLES

Example 1.

Let $X = (0, \frac{1}{2}]$ equipped with the standard metric $d(x, y) = |x - y|$. This is clearly an incomplete metric space. The mapping $T : X \to X$ be defined as $T(x) = x^2$ is a contraction but T has no fixed point.

Example 2.

Let $X = \{x \in \mathbf{R} : x \geq 1\}$ with the standard metric. Let $T : X \to X$ be given by $Tx = x + \frac{1}{x}$. Then T is contractive but T has no fixed point. Note that T is not a contraction.

Example 3.

Let $X = [0, \infty)$ with the standard metric. Let $T : X \to X$ be given by $Tx = \frac{1}{1+x^2}$. Then T is contractive but has no fixed point. Note that T is not a contraction.

Example 4.

Let $T : C[a, b] \to C[a, b]$, $(-\infty < a < b < \infty)$ with uniform norm, be defined as

$$[T(f)](t) = \int_a^t f'(x)dx.$$

Then it can be shown that

$$[T^k(f)](t) = \frac{1}{(k-1)!}\int_a^t (t-x)^{k-1} f(x)dx.$$

For sufficiently large values of k, the mapping T^k is a contraction, whereas T is not a contraction if $(b-a) > 1$.

14.7 SUMMARY

This unit is the presentation of the work related to Fixed Point theory. In this unit in starting the contraction, contractive and non-expansive mappings defined in a simple manner. After the important Banach contraction fixed point theorem defined in a systematic manner. Then different theorems for find the fixed point defined. After that examples are defined.

14.8 GLOSSARY

i. **Metric space:** Let $X \neq \emptyset$ be a set then the metric on the set X is defined as a function $d: X \times X \to [0, \infty)$ such that some conditions are satisfied.

ii. **Vector space: -** Let V be a nonempty set with two operations

a. **Vector addition:** If any $u, v \in V$ then $u + v \in V$

b. **Scalar Multiplication:** If any $u \in V$ and $k \in F$ then $ku \in V$

Then V is called a vector space (over the field F) if the following axioms hold for any vectors if the some conditions hold.

iii. **Normed space:-** Let X be a vector space over scalar field K. A *norm* on a (real or complex) vector space X is a real-valued function on X ($\|x\|: X \to K$) whose value at an $x \in X$ is denoted by $\|x\|$ and which has the four properties here x and y are arbitrary vectors in X and α is any scalar.

iv. **Banach space:-** A complete normed linear space is called a Banach space.

v. Cauchy sequence.

vi. Convergent sequence.

vii. Uniqueness.

viii. Function(mappings).

CHECK YOUR PROGRESS

4. The cosine function is continuous in $[-1, 1]$ and maps it into $[-1, 1]$, and thus must have a fixed point. True/False

5. Every contraction map is discontinuous. True/False

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14.10 TERMINAL QUESTIONS

1. What is an example of a fixed point theory?

……………………………………………..

2. How do you solve for a fixed point?

……………………………………….

3. What are the applications of fixed point?

…………………………………………………

4. State and proof Banach contraction principle.

……………………………………………………..

14.11 ANSWERS

CHECK YOUR PROGRESS

2. No

3. False

4. True

5. False

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