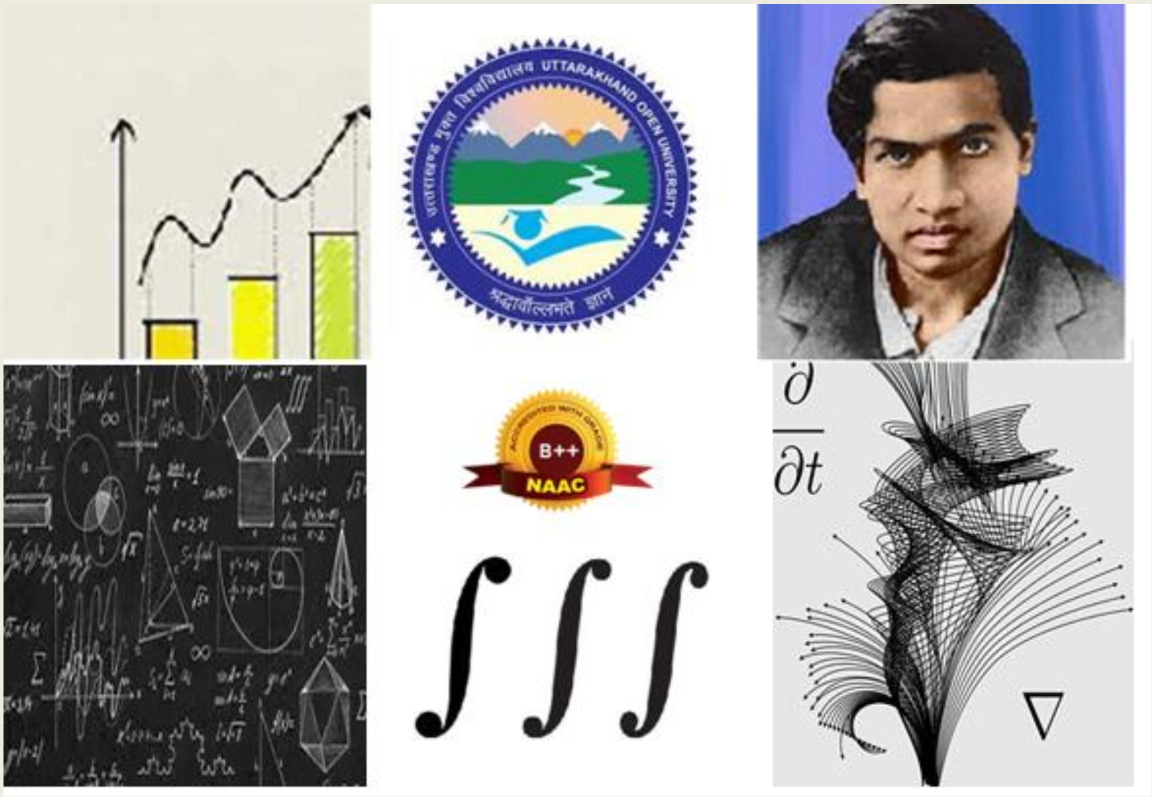
Master of Science

Mathematics

Third Semester

MAT 604
FLUID MECHANICS



DEPARTMENT OF MATHEMATICS
SCHOOL OF SCIENCES
UTTARAKHAND OPEN UNIVERSITY
HALDWANI, UTTARAKHAND
263139

COURSE NAME: FLUID MECHANICS

COURSE CODE: MAT 604

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Uttarakhand Open University
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COURSE INFORMATION

The present self-learning material **"FLUID MECHANICS"** has been designed for M.Sc. (Third Semester) learners of Uttarakhand Open University, Haldwani. This course is divided into 14 units of study. This Self Learning Material is a mixture of Four Block.

First block is **KINEMATICS OF FLUIDS IN MOTION,** in this block Real fluids and Ideal fluids, Newton's law of viscosity, Newtonian and non-Newtonian fluid, Types of non-Newtonian fluids, hypothesis of continuum, Velocity of a Fluid at a point defined clearly.

Second block is **EQUATIONS OF MOTION OF A FLUID,** in this block Pressure at a Point in a Fluid at Rest or in a Motion, Euler's Equation of motion defined clearly.

Third block is **TWO-DIMENSIONAL FLOW,** in this block The Stream Function, Two-Dimensional Image System are defined.

Fourth block **SOURCES, SINKS AND DOUBLETS AND STOKES FUNCTION,** in this block concept Relations between Cartesian Components, Stokes function defined.

Adequate number of illustrative examples and exercises have also been included to enable the leaners to grasp the subject easily.

CONTENTS
MAT - 604

Course Name: FLUID MECHANICS

Course Code: MAT604

BLOCK-I

KINEMATICS OF FLUIDS IN MOTION

UNIT 1: *BASICS OF FLUID*

CONTENTS:

1.1 *INTRODUCTION*

A fluid is a substance that continuously deforms under an applied shear stress, regardless of the magnitude of the stress. Unlike solids, which resist deformation and maintain a fixed shape, fluids encompass both liquids and gases. They are characterized by their ability to flow and conform to the shape of their containers. The molecules within a fluid are free to move past one another, allowing for this fluidity and the transmission of pressure in all directions. This unique property of fluids makes them essential in numerous natural and engineered systems, from the flow of water in rivers to the circulation of air in the atmosphere and the operation of hydraulic machines. Understanding the behavior of fluids is crucial for a wide range of applications, including engineering, meteorology, medicine, and environmental science.

Mechanics is the oldest physical science, focusing on the behavior of both stationary and moving bodies under the influence of forces. The branch of mechanics that addresses bodies at rest is known as statics, while the branch that examines bodies in motion is called dynamics. Fluid mechanics, a subcategory of mechanics, studies the behavior of fluids at rest (fluid statics) or in motion (fluid dynamics) and the interaction of fluids with solids or other fluids at their boundaries. Fluid mechanics is often referred to as fluid dynamics, considering fluids at rest as a special case of motion with zero velocity.

1.2 *OBJECTIVES*

After completion of this unit learners will be able to:

(i) Define the concept of fluid.

(ii) Describe the Newton's law of viscosity.

(iii) Differentiate between Newtonian and non-Newtonian fluid.

(iv) Explain different types of non-Newtonian fluid.

(v) Explain the continuum hypothesis.

1.3 *REAL FLUID AND IDEAL FLUIDS*

1.3.1 VISCOSITY OF FLUIDS

Viscosity is a measure of a fluid's resistance to deformation and flow. It quantifies the internal friction between layers of a fluid as they move relative to each other. A fluid with high viscosity, such as honey, flows slowly and resists motion, while a fluid with low viscosity, like water, flows easily and quickly. Viscosity plays a critical role in fluid dynamics, affecting how fluids move through pipes, around objects, and within various natural and industrial processes. It is influenced by factors such as temperature and pressure; for instance, most fluids become less viscous at higher temperatures.

The viscosity of a liquid decreases significantly with increasing temperature, while the viscosity of a gas increases as the temperature rises. Although viscosity also depends on pressure, this effect is generally minor compared to the influence of temperature in most fluid dynamics problems.

1.3.2 DEFINITON OF REAL AND IDEAL FLUIDS

Real Fluids: Real fluids are those that exhibit viscosity, meaning they resist motion due to internal friction between their layers. This viscosity causes energy loss when the fluid flows, leading to phenomena such as drag and turbulence. Examples of real fluids include water, oil, air, and honey. These fluids display characteristics such as shear stress, heat conduction, and compressibility, making them more complex to analyze in practical applications.

Ideal Fluids: Ideal fluids are hypothetical fluids that are assumed to have no viscosity and are incompressible. They provide a simplified model for fluid dynamics problems, allowing easier analysis and mathematical modeling. Ideal fluids do not resist shear stress, leading to the assumption of no energy loss during flow. While no real fluid perfectly fits this description, the concept of an ideal fluid is useful for understanding fundamental principles and approximations.

1.4 *NEWTONIAN AND NON-NEWTONIAN FLUID*

1.4.1 NEWTON'S LAW OF VISCOSITY

Newton's law of viscosity describes the relationship between the shear stress and the velocity gradient in a fluid. Newton's law of viscosity states that the shear stress (τ) between adjacent fluid layers is directly proportional to the rate of change of velocity $\left(\frac{du}{dy}\right)$ with respect to the distance perpendicular to the direction of flow.

Mathematically, it can be expressed as, $\tau = \mu\left(\frac{du}{dy}\right)$, where μ is a constant of proportionality which is called the coefficient of viscosity or coefficient of dynamic viscosity of the fluid. The components in the Newton's law of viscosity are defined as:

(i) **Shear Stress** (τ): This is the force per unit area exerted by the fluid layers upon each other. It acts tangentially to the surface.

(ii) **Velocity Gradient**$\left(\frac{du}{dy}\right)$: This represents the rate at which the fluid velocity changes with respect to the distance in the direction perpendicular to the flow. A higher velocity gradient indicates a steeper change in velocity between layers.

(iii) **Dynamic Viscosity (μ)**: This is a measure of the fluid's resistance to deformation or flow. It is a property of the fluid that indicates how "thick" or "thin" the fluid is. Higher viscosity means the fluid is thicker and resists flow more.

1.4.2 DEFINITION OF NEWTONIAN AND NON-NEWTONIAN FLUID

Newtonian fluid: Fluids that follow Newton's law of viscosity, where the shear stress is directly proportional to the velocity gradient, are called Newtonian fluids. Common examples include water, air, and most common oils.

Non-Newtonian fluid: Fluids that do not follow Newton's law of viscosity are termed non-Newtonian fluids. In these fluids, the relationship between shear stress and the velocity gradient is nonlinear and can depend on the shear rate, time, or other factors. Examples include ketchup, toothpaste, and blood.

1.4.3 TYPES OF NON-NEWTONIAN FLUIDS

Non-Newtonian fluids exhibit a variety of behaviors that deviate from the linear relationship between shear stress and shear rate observed in Newtonian fluids. Figures 1.1 and 1.2 show that these fluids can be classified into several types based on how their viscosity changes with shear rate, time, or stress. A detailed explanation of the types of non-Newtonian fluids is as follows:

(i) **Dilatant fluid**: This is a shear-thickening fluid, applying a higher shear rate increases the internal resistance, making the fluid more viscous. This behavior is often observed in suspensions where particles crowd together under shear. Examples include mixtures of corn starch or sand in water. A well-known example is quicksand, which solidifies when disturbed.

(ii) **Pseudoplastic fluid**: This is a shear-thinning fluids, applying a higher shear rate reduces the internal resistance, making the fluid flow more easily. This behavior is common in many polymer solutions and biological fluids. Examples include polymer solutions, colloidal suspensions, paper pulp in water, latex paint, blood plasma, and syrup. A classic example is paint, which is thick when poured but becomes thin when brushed with high strain rates.

(iii) **Bingham plastic fluid:** Bingham plastics do not flow until the applied shear stress exceeds a specific yield value. Once this threshold is surpassed, they flow like a viscous fluid. This fluid requires a certain yield stress to start flowing; behaves like a solid until this stress is exceeded. Examples include suspensions of clay, drilling mud, toothpaste, mayonnaise, chocolate, and mustard. A classic example is ketchup, which remains in the bottle until agitated or shaken.

(iv) **Rheopectic fluid**: Rheopectic fluids become more viscous over time when subjected to continuous shear stress. These fluids require a gradually increasing shear stress to maintain a constant strain rate. Examples include lubricants, and printer inks.

(v) **Thixotropic fluid**: Thixotropic fluids become less viscous over time when subjected to continuous shear stress. This behavior is reversible when the shear stress is removed. These fluids requires decreasing stress to maintain a constant strain rate. Examples include yogurt, gels, and clays.

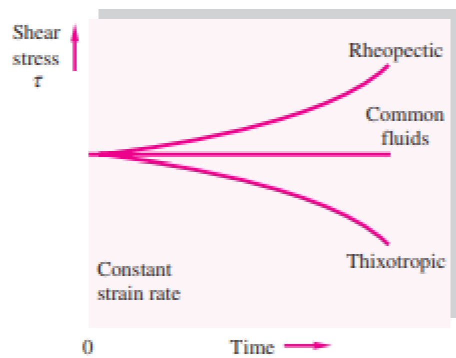

Fig 1.1 Effect of time on applied stress

Ref: Frank M. White (2011), *Fluid Mechanics*, McGraw Hill.

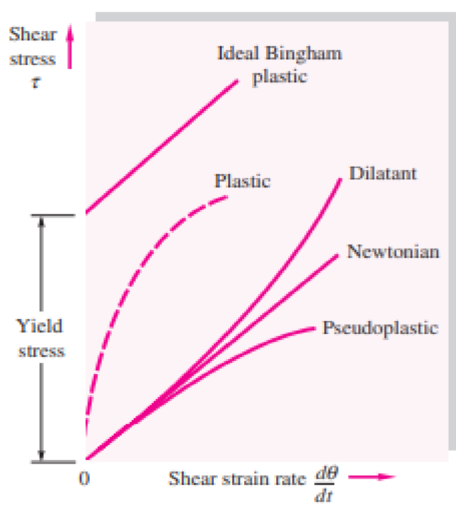

Fig 1.2 Stress versus strain rate

Ref: Frank M. White (2011), *Fluid Mechanics*, McGraw Hill.

1.5 *HYPOTHESIS OF CONTINUUM*

It is widely understood that matter consists of molecules or atoms in constant random motion. In fluid dynamics, analyzing individual molecules isn't practical or necessary for mathematical purposes. Instead, we focus on the macroscopic behavior of fluids, considering them as continuously distributed in space. This assumption is termed the *continuum hypothesis*. Within this continuum concept, we can indefinitely subdivide a fluid element. Additionally, we define a fluid particle as the fluid contained within an infinitesimally small volume.

The continuum hypothesis is justified by the large number of molecules present in macroscopic volumes of fluids. For example, even in a tiny volume of air, there are trillions of molecules interacting with each other. This vast number of molecules allows for the statistical averaging of properties, leading to well-defined macroscopic properties. This hypothesis is applied extensively in fluid mechanics to simplify the governing equations of fluid flow. The Navier-Stokes equations, which describe the motion of fluids, are formulated based on the continuum assumption. These equations treat fluid properties as continuous fields and are widely used in engineering and scientific simulations to predict fluid behavior in various contexts, from aerodynamics and hydrodynamics to chemical engineering processes.

1.6 *VELOCITY OF FLUID AT A POINT*

1.6.1 VELOCITY OF FLUID PARTICLE

Let the fluid particle be at P at any time t and let it be at Q at time $t+\delta t$ such that

$$\overrightarrow{OP}=\mathbf{r} \text{ and } \overrightarrow{OQ}=\mathbf{r}+\delta\mathbf{r}$$

Then in the interval δt the movement of the particle is $\overrightarrow{PQ}=\delta\mathbf{r}$ and hence the velocity of the fluid particle $\mathbf{q}$ at P is given by

$$\mathbf{q}=\lim_{\delta t\to 0}(\delta\mathbf{r}/\delta t)=d\mathbf{r}/dt,$$

assuming such a limit to exist uniquely. Taking the fluid as continuous, the above assumption is justified. Clearly $\mathbf{q}$ is a function of $\mathbf{r}$ and t and hence it can be expressed as $\mathbf{q}=f(\mathbf{r},t)$. If u,v,w are the components of $\mathbf{q}$ along the axes, we have

$$\mathbf{q}=u\mathbf{i}+v\mathbf{j}+w\mathbf{k}$$

1.6.2 ACCELERATION OF FLUID PARTICLE

Suppose a fluid particle moves from $P(x,y,z)$ at time t to $Q(x+\delta x,y+\delta y,z+\delta z)$ at time $t+\delta t$. Let

$$\mathbf{q}=(u,v,w)=u\mathbf{i}+v\mathbf{j}+w\mathbf{k} \qquad (1.1)$$

be the velocity of the fluid particle at P and let $\boldsymbol{q}+\delta\boldsymbol{q}$ be the velocity of the same fluid particle at Q. Then, we have

$$\delta\boldsymbol{q}=\frac{\partial\boldsymbol{q}}{\partial x}\delta\boldsymbol{x}+\frac{\partial\boldsymbol{q}}{\partial y}\delta y+\frac{\partial\boldsymbol{q}}{\partial z}\delta z+\frac{\partial\boldsymbol{q}}{\partial t}\delta t \text{ or } \frac{\delta\boldsymbol{q}}{\delta t}$$
$$=\frac{\partial\boldsymbol{q}}{\partial x}\frac{\delta\boldsymbol{x}}{\delta t}+\frac{\partial\boldsymbol{q}}{\partial y}\frac{\delta y}{\delta t}+\frac{\partial\boldsymbol{q}}{\partial z}\frac{\delta z}{\delta t}+\frac{\partial\boldsymbol{q}}{\partial t} \qquad (1.2)$$

Let

$$\left.\begin{aligned} \lim_{\delta t\to 0}\frac{\delta \mathbf{q}}{\delta t}=\frac{D\mathbf{q}}{Dt} \text{ or } \frac{d\mathbf{q}}{dt}, \quad &\text{and} \quad \lim_{\delta t\to 0}\frac{\delta x}{\delta t}=\frac{dx}{dt}=u \\ \lim_{\delta t\to 0}\frac{\delta y}{\delta t}=\frac{dy}{dt}=v \quad &\quad \lim_{\delta t\to 0}\frac{\delta z}{\delta t}=\frac{dz}{dt}=w \end{aligned}\right\} \qquad (1.3)$$

Making $\delta t\to 0$ and using (1.3), (1.2) reduces to

$$\boldsymbol{a}=\frac{D\mathbf{q}}{Dt}=u\frac{\partial \mathbf{q}}{\partial x}+v\frac{\partial \mathbf{q}}{\partial y}+w\frac{\partial \mathbf{q}}{\partial z}+\frac{\partial \mathbf{q}}{\partial t} \qquad (1.4)$$

Let

$$\nabla=(\partial/\partial x)\mathbf{i}+(\partial/\partial y)\mathbf{j}+(\partial/\partial z)\mathbf{k} \qquad (1.5)$$

From (1.1) and (1.5),

$$\mathbf{q}\cdot\nabla=u(\partial/\partial x)+v(\partial/\partial y)+w(\partial/\partial z) \qquad (1.6)$$

Using (1.6), (1.4) may be re-written as

$$\boldsymbol{a}=\frac{D\mathbf{q}}{Dt}=(\mathbf{q}\cdot\nabla)\mathbf{q}+\frac{\partial \mathbf{q}}{\partial t}, \qquad (1.7)$$

which shows that the acceleration a of a fluid particle of fixed identity can be expressed as the material derivative of the velocity vector $\mathbf{q}$.

Hence, the components of acceleration in Cartesian coordinates (x, y, z) can be written as $\boldsymbol{a}=a_x\mathbf{i}+a_y\mathbf{j}+a_z\mathbf{k}$. Then (1.4) yields:

$$a_x\mathbf{i}+a_y\mathbf{j}+a_z=u\frac{\partial}{\partial x}(u\mathbf{i}+v\mathbf{j}+w\mathbf{k})+v\frac{\partial}{\partial y}(u\mathbf{i}+v\mathbf{j}+w\mathbf{k})$$
$$+w\frac{\partial}{\partial z}(u\mathbf{i}+v\mathbf{j}+w\mathbf{k})+\frac{\partial}{\partial \imath}(u\mathbf{i}+v\mathbf{j}+w\mathbf{k})$$

$$a_x=\frac{Du}{Dt}=u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+\frac{\partial u}{\partial t},$$

$$a_y=\frac{Dv}{Dt}=u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+\frac{\partial v}{\partial t},$$
$$a_z=\frac{Dw}{Dt}=u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}+\frac{\partial w}{\partial t}.$$

1.6.3 EXAMPLES BASED ON VELOCITY AND ACCELERATION

Example 1. If the velocity distribution is $\mathbf{q} = \mathbf{i}Ax^2y + \mathbf{j}By^2zt + \mathbf{k}Czt^2$, where A, B, C, are

constants, then find the acceleration and velocity components.

Solution. The acceleration $\boldsymbol{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ is given by

$$\boldsymbol{a} = \frac{\partial \mathbf{q}}{\partial t} + u\frac{\partial \mathbf{q}}{\partial x} + v\frac{\partial \mathbf{q}}{\partial y} + w\frac{\partial \mathbf{q}}{\partial z}$$

Also

$$\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} = \mathbf{i}Ax^2y + \mathbf{j}By^2zt + \mathbf{k}Czt^2$$

Hence,

$$u = Ax^2y,\ v = By^2zt,\ w = Czt^2$$

Hence, the expression for acceleration becomes:

$$\begin{aligned}\boldsymbol{a} &= By^2z\mathbf{j} + 2Czt\mathbf{k} + Ax^2y \times (2\,\mathrm{A}xy\mathbf{i}) + By^2zt(Ax^2\mathbf{i} + 2Byzt\mathbf{j}) + Czt^2(By^2t\mathbf{j} + Ct^2\mathbf{k}) \\ &= A(2Ax^3y^2 + Bx^2y^2zt)\mathbf{i} + B(y^2z + 2By^3z^2t^2 + Cy^2zt^3)\mathbf{j} + C(2zt + Czt^4)\mathbf{k}\end{aligned}$$

The components of the acceleration (a_x, a_y, a_z) are given by

$$a_x = A(2Ax^3y^2 + Bx^2y^2zt),\ a_y = B(y^2z + 2By^3z^2t^2 + Cy^2zt^3),\ a_z = C(2zt + Czt^4)$$

Example 2. Determine the acceleration at the point (2,1,3) at $t = 0.5$sec, if $u = yz + t$, $v = xz - t$ and $w = xy$.

Solution. Velocity field $\mathbf{q}$ at the point (x, y, z) is given by

$$\mathbf{q} = \boldsymbol{u}\mathbf{i} + v\mathbf{j} + w\mathbf{k} = (yz + t)\mathbf{i} + (xz - t)\mathbf{j} + xy\mathbf{k}.$$

The acceleration $\boldsymbol{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ is given by

$$\begin{aligned}\boldsymbol{a} &= \partial\boldsymbol{q}/\partial t + u(\partial\boldsymbol{q}/\partial t) + v(\partial\boldsymbol{q}/\partial t) + w(\partial\boldsymbol{q}/\partial t) \\ &= (\mathbf{i} - \mathbf{j}) + (yz + t)(z\mathbf{j} + y\mathbf{k}) + (xz - t)(z\mathbf{i} + x\mathbf{k}) + xy(y\mathbf{i} + x\mathbf{j}) \\ &= (1 + xz^2 + xy^2 - tz)\mathbf{i} + (-1 + yz^2 + x^2z + zt)\mathbf{j} + (y^2z + x^2z + yt - xt)\mathbf{k}\end{aligned}$$

Acceleration at (2,1,3) at $t = 0.5$ is given by $\boldsymbol{a} = 19.5\mathbf{i} + 13.5\mathbf{j} + 6.5\mathbf{k}$

Hence the components a_x, a_y, a_z of acceleration are given by

$$a_x = 19.5,\ a_y = 13.5 \text{ and } a_z = 6.5$$

1.7 *SUMMARY*

This unit explains the following topics:

(i) Definition of Fluid.

(ii) Real and Ideal Fluids.

(iii) Definition of Newtonian and non-Newtonian fluids based on Newton's law of viscosity.

(iv) Different types of non-Newtonian fluids.

(v) Continuum hypothesis.

(vi) Velocity and acceleration of fluid particle.

1.8 *GLOSSARY*

(i) Fluid

(ii) Viscosity

(iii) Newton's law of viscosity

(iv) Newtonian and non-Newtonian fluids

(v) Shear stress

1.9 *REFERENCES AND SUGGESTED READINGS*

(i) M. D. Raisinghanai (2013), *Fluid Dynamics*, S. Chand & Company Pvt. Ltd.

(ii) Frank M. White (2011), *Fluid Mechanics*, McGraw Hill.

(iii) John Cimbala and Yunus A Çengel (2019), *Fluid Mechanics: Fundamentals and Applications*, McGraw Hill.

(iv) P.K. Kundu, I.M. Cohen & D.R. Dowling (2015), Fluid Mechanics, Academic Press; 6th edition.

(v) F.M. White & H. Xue (2022), Fluid Mechanics, McGraw Hill; Standard Edition.

(vi) S.K. Som, G. Biswas, S. Chakraborty (2017), Introduction to Fluid Mechanics and Fluid Machines, McGraw Hill Education; 3rd edition.

1.10 *TERMINAL QUESTIONS*

1. What is a fluid?

2. What is Newton's law of viscosity?

3. Define Newtonian and non-Newtonian fluids.

4. What is Continuum hypothesis for fluid?

5. Explain different types of non-Newtonian fluids?

6. Determine the acceleration of a fluid particle from the following flow field:

$\mathbf{q} = \mathbf{i}Ax^2yt + \mathbf{j}Bx^2yt + \mathbf{k}Cxyz.$

Solution: $a_x = A(\mathrm{x}y^2 + A\mathrm{x}y^4t + 2Bx^3y^2t^2),\ a_y = B(x^2y + 2\mathrm{A}x^2y^3\mathrm{t} + Bx^4yt^2),\ a_z = C(\mathrm{A}xy^3z + \mathrm{B}x^3\mathrm{yzt} + zx^2y^2)$

UNIT 2: *STREAM LINES*

CONTENTS:

2.1 *INTRODUCTION*

Fluid mechanics is branch of mathematical science which deals with study of fluids, in both stages, in motion or at rest. Fluids are classified in forms of liquids and gases. The liquids have relatively close molecules with cohesive attraction. Generally, liquids are having constant volume and free surface in a gravitational field. While, the gases have wide spaced molecules with very less cohesive attraction. The volume of gas is not definite and also not disturbed by gravitational effects.

The fluid mechanics is significantly applicable in engineering, sciences and human activities as 75% part of earth is occupied by water and 100% by air. The applications include medical studies blood flow, aerodynamics, hydrodynamics oceanography, hydrology and energy generation etc.

The practical applications of fluid flow, not scientifically acknowledged, were shown in ancient civilizations like water supply, drainage, irrigation systems, design of boats and arrows etc. Later on around 246 BC, Archimedes stated that the body immersed in a fluid is buoyed up by a force equal to the weight of the fluid, which is displaced by body. It is often known as Archimedes' principle. In Fifteenth century, Leonardo Da Vinci observed movements of water like hydraulic jumps, eddy formation and analyzed complex motions in terms of linear and circular components. Due to lack of mathematical tools, these observations were prevented from set of generally applicable laws, which were formulated by Navier and Stokes in nineteenth century.

This unit deals with stream lines, path lines, steady and unsteady flows, velocity potential and vorticity vectors. We also discuss a number of examples and ways to evaluate these physical quantities.

2.2 *OBJECTIVES*

After completion of this unit learners will be able to

i. Define the concept stream lines, path lines, velocity potential and vorticity vectors.
ii. Describe the difference and relation between stream lines and path lines.
iii. Explain the steady and unsteady flows.
iv. Understand the uniform and non-uniform flows.

2.3 *STREAM LINES*

The stream line is defined as flow line or curve traced in fluid such that the tangent at its any point indicates the direction of motion or fluid velocity at that point.

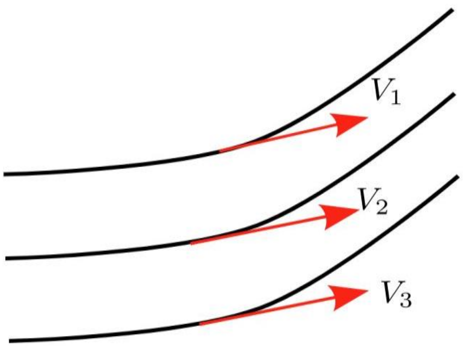

Figure 2

The stream line can also be defined as a line which is parallel to velocity vector at every point. Let $\vec{s} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ be the positon vector at any point *A* along the streamline and let $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$ be the velocity vector at same point *A*. Then, $\vec{v}$ and $\overrightarrow{ds}$ are parallel at point *A*. Thus, we have

$$\vec{v} \times \overrightarrow{ds} = 0$$

Then, $$(v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}) \times \left(dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}\ \right) = 0$$

$$(v_2dz - v_3dy)\hat{\imath} + (v_3dx - v_1dz)\hat{\jmath} + (v_1dy - v_2dx)\hat{k} = 0$$

Equating the components of both sides, it provides three differential equations

$$v_2dz - v_3dy = 0, \qquad v_3dx - v_1dz = 0, \qquad v_1dy - v_2dx = 0$$

On solving above equations, one gets the equation of streamline

$$\frac{dx}{v_1} = \frac{dy}{v_2} = \frac{dz}{v_3} \qquad \ldots(1)$$

The integration of equation (1) gives family of curves. To fix the integrating constants, it is required to define some points from where the stream line passes.

For two dimensional spaces, it has $dz = 0$ and $v_3 = 0$. Therefore, the equation of stream line is

$$\frac{dx}{v_1} = \frac{dy}{v_2} \qquad \ldots(2)$$

This is an ordinary differential equation. Also, the integration of equation (2) gives family of curves. To fix the integrating constants, it is required to define some points from where the stream line passes.

Remarks: 1. There exists unique stream line at a point if all v_1, v_2 and v_3 not equal to zero.

2. If the velocity vectors are zero at a point then singularities exist and the point is called critical point.

2.4 *STREAM TUBES*

If (x^0, y^0, z^0) is any set of points in closed curve in fluid then stream lines which passes through all these points of closed curve form a surface of tubes, called stream tubes. As there is no fluid flow across the surface, therefore end point of stream tube transfers the equal mass flow and stream tube is treated like channel via fluid is flowing.

2.5 *PATH LINE*

The curve or path or trajectory along which an individual fluid particle travels during its motion, is called path line. The direction of path is obtained by streamlines at each point over a certain period.

Let $\vec{s} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ be the positon vector at any point *A* and let $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$ be the velocity vector at same point *A*. Then, the path line are defined by differential equation

$\frac{d}{dt}(\vec{s}) = \vec{v}$ i.e.

$\frac{dx}{dt} = v_1$, $\frac{dy}{dt} = v_2$ and $\frac{dz}{dt} = v_3$.

Note: If at a fixed time $t = t_0$, the position of fluid particle is (x^0, y^0, z^0) then path line can be derived from following equations

$\frac{dx}{dt} = v_1$, $\frac{dy}{dt} = v_2$ and $\frac{dz}{dt} = v_3$

with initial condition

$x(t_0) = x^0$, $y(t_0) = y^0$ and $z(t_0) = t^0$.

➢ Differences between streamlines and path lines

1. Streamline represents the direction of each fluid particle at given instant of time while path line represents the path of individual fluid particle at each instant and may be different for each particle.

2. Streamline shows fluid flow at a given instant and can be used to visualize the overall flow pattern. However, path line gives the entire history of movement for individual fluid particle.

➢ Relation between stream lines and path lines

For the steady motion, stream lines and path lines are identical.

CHECK YOUR PROGRESS

The equation of streamline follows

(a) $\vec{v} \times \overrightarrow{ds} = 0$

(b) $\vec{v} \cdot \overrightarrow{ds} = 0$

(c) $\overrightarrow{ds} \cdot \vec{v} = 0$

(d) None of these

Ans: (a)

2.6 *DIFFERENT TYPES OF FLOWS*

2.6.1 UNIFORM FLOW

If all the fluid particles in a fluid flow moves with same velocity at each point of cross section then flow is called uniform flow. In uniform flow, the velocity is function of time only.

2.6.2 NON-UNIFORM FLOW

If the fluid particles in a fluid flow do not move with same velocity and change from one point to another then flow is called non-uniform flow. In non-uniform flow, the velocity may change with position.

2.6.3 STEADY FLOW

A fluid flow in which the flow pattern is independent of time is called steady flow i.e. properties like pressure, velocity, temperature etc. remain unchanged with time.

2.6.4 UNSTEADY FLOW

A fluid flow in which the flow pattern depends on time is called unsteady flow i.e. properties like pressure, velocity, temperature etc. change with time.

2.7 *VELOCITY POTENTIAL*

Let $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$ be the velocity vector at a time t in an incompressible fluid then a scalar function $\emptyset$ of space and time is called velocity potential if it follows

$$\vec{v} = -\nabla\emptyset = -grad\ \emptyset$$

i.e.

$$v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k} = -\left(\frac{\partial\emptyset}{\partial x}\hat{\imath} + \frac{\partial\emptyset}{\partial y}\hat{\jmath} + \frac{\partial\emptyset}{\partial z}\hat{k}\right)$$

Or $\quad v_1 = -\frac{\partial\emptyset}{\partial x}, \qquad v_2 = -\frac{\partial\emptyset}{\partial y} \quad$ and $\quad v_3 = -\frac{\partial\emptyset}{\partial z}$.

Remarks: 1. The necessary and sufficient condition for velocity potential is that flow should be irrotational i.e.

$$\nabla \times \vec{v} = 0$$

0r $\left(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z}\right)\hat{\imath}+\left(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x}\right)\hat{\jmath}+\left(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right)\hat{k}=0.$

2. If the velocity potential $\emptyset$ exists, then it satisfies Laplace equation i.e.

$$\frac{\partial^2\emptyset}{\partial x^2}+\frac{\partial^2\emptyset}{\partial y^2}+\frac{\partial^2\emptyset}{\partial z^2}=0.$$

2.8 *VORTICITY VECTOR*

Let $\vec{v}=v_1\hat{\imath}+v_2\hat{\jmath}+v_3\hat{k}$ be fluid velocity then the vector Ω is called vorticity vector and defined as

$$\Omega=\nabla\times\vec{v}\,.$$

i.e. $\Omega=\left(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z}\right)\hat{\imath}+\left(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x}\right)\hat{\jmath}+\left(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right)\hat{k}.$

The components Ω_x , Ω_y and Ω_z of vorticity vector are

$$\Omega_x\hat{\imath}+\Omega_y\hat{\jmath}+\Omega_z\hat{k}=\left(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z}\right)\hat{\imath}+\left(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x}\right)\hat{\jmath}+\left(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right)\hat{k}$$

Therefore

$$\Omega_x=\left(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z}\right),\quad \Omega_y=\left(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x}\right)\quad \text{and}\quad \Omega_z=\left(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right).$$

2.9 SOLVED EXAMPLES

Example 1: Determine the equation of stream lines for the velocity vector $v=x\,\hat{\imath}+y\,\hat{\jmath}$ passing through point (2,1).

Solution: The given velocity vector is

$$v=x\,\hat{\imath}+y\,\hat{\jmath}.$$

Comparing the above vector with $v = v_1\hat{\imath} + v_2\hat{\jmath}$, we have

$v_1 = x$ and $v_2 = y$.

Then, equation of stream lines is

$\frac{dx}{v_1} = \frac{dy}{v_2}$ therefore $\frac{dx}{x} = \frac{dy}{y}$.

On solving above system, we have

$\frac{dx}{x} = \frac{dy}{y}$ i.e. $\log x - \log y = \log c$ i.e. $\frac{x}{y} = c$ ...(1)

where c is constant of integration.

As, the stream lines passes through point (2,1), therefore $c = 2$.

Thus, the stream line is $x - 2y = 0$.

Example 2: Determine the equation of stream lines passing through (1,2,1) for the velocity vector $v = x\,\hat{\imath} - 2y\,\hat{\jmath} + 2z\hat{k}$.

Solution: The given velocity vector is

$v = x\,\hat{\imath} - 2y\,\hat{\jmath} + 2z\hat{k}$.

Comparing the above vector with $v = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$, we have

$v_1 = x$, $v_2 = -2y$ and $v_3 = 2z$.

Then, equation of stream lines is

$\frac{dx}{v_1} = \frac{dy}{v_2} = \frac{dz}{v_3}$ therefore $\frac{dx}{x} = \frac{dy}{-2y} = \frac{dz}{2z}$.

On solving above system, we have

$\frac{dx}{x} = \frac{dy}{-2y}$ i.e. $\log x + log\, y^2 = \log c_1$ i.e. $x\,y^2 = c_{1.}$ .... (1)

And, $\frac{dx}{x} = \frac{dz}{2z}$ i.e. $\log x - log\, z^2 = \log c_2$ i.e. $\frac{x}{z^2} = c_2$. .... (2)

Thus, the stream lines are provided by intersection of (1) and (2), where c_1 and c_2 are constants of integration. As, the streamlines passes through point (1,2,1) therefore $c_1 = 4$ and $c_2 = 1$.

Thus, the required stream line is provided by interaction of curves $x\,y^2 = 4$ and $\frac{x}{z^2} = 1$.

Example 3: Find the stream lines for the flow whose velocity components are $v_1 = \frac{xt}{1+t^2}$, $v_2 = \frac{2yt}{1+t^2}$ and $v_3 = \frac{zt}{1+t^2}$.

Solution: The given velocity components are

$$v_1 = \frac{xt}{1+t^2}, \qquad v_2 = \frac{2yt}{1+t^2} \quad \text{and} \quad v_3 = \frac{zt}{1+t^2}.$$

Then, equation of stream lines is

$$\frac{dx}{v_1} = \frac{dy}{v_2} = \frac{dz}{v_3} \qquad \text{therefore} \qquad \frac{dx}{\left(\frac{xt}{1+t^2}\right)} = \frac{dy}{\left(\frac{2yt}{1+t^2}\right)} = \frac{dz}{\left(\frac{zt}{1+t^2}\right)} .$$

On solving above system, we have

$$\frac{dx}{\left(\frac{xt}{1+t^2}\right)} = \frac{dy}{\left(\frac{2yt}{1+t^2}\right)} \quad \text{i.e.} \quad \log x - \log y^2 = \log c_1 \quad \text{i.e.} \quad \frac{x}{y^2} = c_1. \quad ...(1)$$

And, $$\frac{dx}{\left(\frac{xt}{1+t^2}\right)} = \frac{dz}{\left(\frac{zt}{1+t^2}\right)} \quad \text{i.e.} \quad \log x - \log z = \log c_2 \quad \text{i.e.} \quad \frac{x}{z} = c_2.$$

…(2)

Thus, the stream lines are provided by intersection of (1) and (2), where c_1 and c_2 are constants of integration.

Example 4: Find the stream lines for the flow whose velocity components are $v_1 = \frac{x}{1+t}$, $v_2 = y$ and $v_3 = 0$.

Solution: The given velocity components are

$$v_1 = \frac{x}{1+t}, \qquad v_2 = y \quad \text{and} \quad v_3 = 0.$$

Then, equation of stream lines is

$\frac{dx}{v_1} = \frac{dy}{v_2} = \frac{dz}{v_3}$ therefore $\frac{dx}{\left(\frac{x}{1+t}\right)} = \frac{dy}{y} = \frac{dz}{0}$.

On solving above system, we have

$\frac{dx}{\left(\frac{x}{1+t}\right)} = \frac{dy}{y}$ i.e. $(1 + \text{t})\log x - \log y = \log c_1$ i.e. $\frac{x^{(1+t)}}{y} = c_1$

...(1)

And, $dz = 0$ i.e. $z = c_2$ ...(2)

Thus, the stream lines are provided by intersection of (1) and (2), where c_1 and c_2 are constants of integration.

Example 5: Determines the path lines for the flow whose velocity components are $v_1 = \frac{2x}{3-t}$, $v_2 = \frac{y}{1+t}$ and $v_3 = \frac{z}{2+t}$.

Solution: The given velocity components are

$v_1 = \frac{2x}{3-t}$, $v_2 = \frac{y}{1+t}$ and $v_3 = \frac{z}{2+t}$.

Then, equation of path lines is

$\frac{dx}{dt} = v_1$, $\frac{dy}{dt} = v_2$ and $\frac{dz}{dt} = v_3$.

On solving above system, we have

$\frac{dx}{dt} = \frac{2x}{3-t}$ i.e. $\frac{dx}{2x} = \frac{dt}{3-t}$ i.e. $\log x - \log(3-t)^2 = \log c_1$
i.e. $x = c_1(3-t)^2$,

$\frac{dy}{dt} = \frac{y}{1+t}$ i.e. $\frac{dy}{y} = \frac{dt}{1+t}$ i.e. $\log y - \log(1+t) = \log c_2$
i.e. $y = c_2(1+t)$,

$\frac{dz}{dt} = \frac{z}{2+t}$ i.e. $\frac{dz}{z} = \frac{dt}{2+t}$ i.e. $\log z - \log(2+t) = \log c_3$ i.e.
$z = c_3(2+t)$,

Thus, the curves $x = c_1(3-t)^2$, $y = c_2(1+t)$ and $z = c_3(2+t)$ provides path lines, where c_1, c_2 and c_3 are constants of integration.

Example 6: Determines the path lines for the flow whose velocity field is given as $= (2xt,\ \frac{y}{t},\ 0)$.

Solution: The given velocity components are

$$v_1 = 2xt\ , \quad v_2 = \frac{y}{t} \quad \text{and} \quad v_3 = 0\ .$$

Then, equation of path lines is

$$\frac{dx}{dt} = v_1, \quad \frac{dy}{dt} = v_2 \quad \text{and} \quad \frac{dz}{dt} = v_3.$$

On solving above system, we have

$\frac{dx}{dt} = 2xt$ i.e. $\frac{dx}{x} = 2t\ dt$ i.e. $\log x - t^2 = \log c_1$ i.e. $x = c_1 e^{t^2}$,

$\frac{dy}{dt} = \frac{y}{t}$ i.e. $\frac{dy}{y} = \frac{dt}{t}$ i.e. $\log y - \log t = \log c_2$ i.e. $y = c_2 t$,

$\frac{dz}{dt} = 0$ i.e. $dz = 0$ i.e. $z = c_3$.

Thus, the curves $x = c_1 e^{t^2}$, $y = c_2 t$ and $z = c_3$ provides path lines, where c_1, c_2 and c_3 are constants of integration.

Example 7: Determines the path lines for the flow whose velocity components are $v_1 = \frac{2xt}{t^2-1}$, $v_2 = \frac{yt}{t^2+1}$ and $v_3 = \frac{z}{t-2}$.

Solution: The given velocity components are

$$v_1 = \frac{2xt}{t^2-1}, \quad v_2 = \frac{yt}{t^2+1} \quad \text{and} \quad v_3 = \frac{z}{t-2}.$$

Then, equation of path lines is

$$\frac{dx}{dt} = v_1, \quad \frac{dy}{dt} = v_2 \quad \text{and} \quad \frac{dz}{dt} = v_3.$$

On solving above system, we have

$\frac{dx}{dt} = \frac{2xt}{t^2-1}$ i.e. $\frac{dx}{x} = \frac{2t\,dt}{t^2-1}$ i.e. $\log x - \log(t^2 - 1) = \log c_1$

i.e. $x = c_1(t^2 - 1)$,

$\frac{dy}{dt} = \frac{yt}{t^2+1}$ i.e. $\frac{dy}{y} = \frac{t\,dt}{t^2+1}$ i.e. $\log y - \frac{1}{2}\log(t^2 + 1) = \log c_2$

i.e. $y = c_2(t^2 + 1)^{\frac{1}{2}}$,

$\frac{dz}{dt} = \frac{z}{t-2}$ i.e. $\frac{dz}{z} = \frac{dt}{t-2}$ i.e. $\log z - \log(t - 2) = \log c_3$ i.e. $z = c_3(t - 2)$

Thus, the curves $x = c_1(t^2 - 1)$, $y = c_2(t^2 + 1)^{\frac{1}{2}}$ and $z = c_3(t - 2)$ provides path lines, where c_1, c_2 and c_3 are constants of integration.

Example 8: If the velocity potential $\emptyset = 2(x^2 + 2xz + y^3)$, then find the velocity components.

Solution: The velocity potential for three - dimensional flow is

$\emptyset = 2(x^2 + 2xz + y^3)$.

Then, the velocity components v_1, v_2 and v_3 are

$v_1 = -\frac{\partial\emptyset}{\partial x}$, $v_2 = -\frac{\partial\emptyset}{\partial y}$ and $v_3 = -\frac{\partial\emptyset}{\partial z}$.

Therefore,

$v_1 = -4(x + z)$, $v_2 = -6y$ and $v_3 = -4x$.

Example 9: If the velocity vector is given by $v = 2x^2yz\hat{\imath} + axz\hat{\jmath} + bxy^2z\hat{k}$, then find the vorticity components.

Solutions: The given velocity vector is

$$v = 2x^2yz\hat{\imath} + axz\hat{\jmath} + bxy^2z\hat{k}.$$

Comparing the above vector with $v = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$, we have

$v_1 = 2x^2yz$, $v_2 = axz$ and $v_3 = bxy^2z$.

Then, the vorticity components are

$\Omega_x = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right), \quad \Omega_y = \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)$ and $\Omega_z = \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right).$

Therefore,

$\Omega_x = 2bxyz - ax, \qquad \Omega_y = 2x^2y - by^2z, \qquad \Omega_z = az - 2x^2z.$

Example 10: Determine the vorticity vector If the velocity vector is given by $v = (axy^2 + z)\hat{\imath} + (by + xz^2)\hat{\jmath} + (x + cyz^2)\hat{k}.$

Solution: The given velocity vector is

$$v = (axy^2 + z)\hat{\imath} + (by + xz^2)\hat{\jmath} + (x + cyz^2)\hat{k}.$$

Comparing the above vector with $v = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$, we have

$v_1 = axy^2 + z, \qquad v_2 = by + xz^2$ and $v_3 = cx + yz^2.$

Then, the vorticity vector is

$\Omega = \Omega_x\hat{\imath} + \Omega_y\hat{\jmath} + \Omega_z\hat{k}$

where

$\Omega_x = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right), \quad \Omega_y = \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)$ and $\Omega_z = \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right).$

Therefore,

$\Omega_x = z^2 - 2xz, \qquad \Omega_y = 1 - c$ and $\Omega_z = z^2 - 2axy.$

Thus, the vorticity vector is

$\Omega = (z^2 - 2xz)\hat{\imath} + (1 - c)\hat{\jmath} + (z^2 - 2axy)\hat{k}$.

2.10 *SUMMARY*

This unit explains

1. Definitions of streamlines, path lines and stream tubes with examples.
2. Relations and differences between streamlines and path lines.
3. Definitions of uniform flow, non-uniform flow, steady flow, unsteady flow, velocity potential, vorticity vector with examples.
4. Necessary and sufficient condition for velocity potential.

2.11 *GLOSSARY*

i. Fluid Flow
ii. Density
iii. Volume
iv. Pressure

2.12 *REFERENCES AND SUGGESTED READINGS*

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2.13 *TEWRMINAL QUESTIONS*

1. The stream lines and path lines are identical if

(a) Flow is steady

(b) Flow is uniform

(c) Flow velocities do not change steadily with time

(d) Flow is neither steady nor uniform

2. The flow in which the velocity of fluid particle at a given instant changes is called

(a) Uniform flow

(b) Non-Uniform flow

(c) Compressible Flow

(d) Incompressible flow

3. The stream line represents

(a) trajectory of fluid particle during its motion

(b) curve in fluid such that the tangent at its any point indicates the direction of motion

(c) velocity at any point

(d) Nature of fluid.

4. If the velocity at any point does not change with time, then flow is

(a) Laminar

(b) Non-uniform

(c) Steady

(d) Unsteady

5. If the motion is irrotational the vorticity vector is……………

6. Which of the following function is valid for velocity potential ∅ ?

(a) $2x^2y$

(b) $x^2 - y$

(c) $x^2 + y$

(d) $x^2 - y^2$

Hint: The velocity potential satisfies Laplace equation $\frac{\partial^2 \emptyset}{\partial x^2} + \frac{\partial^2 \emptyset}{\partial y^2} + \frac{\partial^2 \emptyset}{\partial z^2} = 0$

7. In a three-dimensional fluid flow, if a velocity potential ∅ exists if fluid flow is

(a) Steady Compressible.

(b) Steady laminar incompressible.

(c) Turbulent incompressible.

(d) Incompressible and Irrotational.

8. If the function $\emptyset = 2ax^3 - xy^2$ represents the velocity potential the value of a is

(a) 6

(b) -6

(c) $\frac{1}{6}$

(d) $-\frac{1}{6}$

9. True/false

(a) The velocity potential exists for rotational motion.

(b) The velocity potential exists for irrotational motion.

(c) If the motion is irrotational then vorticity vector is 1.

(d) If the motion is irrotational then vorticity vector is 0.

10. The velocity components for incompressible fluid are $(x, 2y, -2z)$, find the equation of stream line passing through (1,2,2).

11. The velocity components for two dimensional fluid flow are $v_1 = e^x$ and $v_2 = \sinh y$, find the equation of stream line.

12. Determine the equation of stream lines passing through (1,2,2) for the velocity vector $v = x^2\,\hat{\imath} - y\,\hat{\jmath} + 2z^2\hat{k}$.

13. Find the streamlines for the flow whose velocity components are $v_1 = x - t + 1$ and $v_2 = -y - t + 2$.

14. If the velocity is constant everywhere, show that the stream lines are straight lines.

15. Determines the path lines for the flow whose velocity components are $v_1 = \frac{x}{1+2t}$, $v_2 = \frac{y^2}{2-t}$ and $v_3 = \frac{5z}{3+t}$.

16. Determines the path lines for the flow whose velocity field is given as $= \left(xt,\ \frac{y}{t^2},\ z\right)$.

17. Determines the path lines for the flow whose velocity components are $v_1 = \frac{xt}{t^2+1}$, $v_2 = \frac{3yt}{t^2-1}$ and $v_3 = \frac{z^2}{1+2t}$.

18. If the velocity vector is given by $v = 2axz\hat{\imath} + 3bxy^2z\hat{\jmath} + 2cx^2z\hat{k}$, then find the vorticity component.

19. Determine the vorticity vector If the velocity vector is given by $v = (xy^2 - 2z)\hat{\imath} + (3y + xz^2)\hat{\jmath} + (5cx + xy^2z)\hat{k}$.

20. Distinguish between stream lines and path lines.

2.14 *ANSWERS*

1. (a)

2. (b)

3. (b)

4. (c)

5. 0

6. (d)

7. (d)

8. (c)

9 (a) F, (b) T, (c) F, (d) T

UNIT 3: *EQUATION OF CONTINUITY*

CONTENTS:

3.1 *INTRODUCTION*

The transfer of various quantities, such as fluid or gas, is described by the continuity equation. The formula describes how a fluid conserves mass while moving. How a fluid conserves mass while moving is described by the equation. The continuity equations can be used to demonstrate the conservation of a wide range of physical phenomena, including energy, mass, momentum, natural numbers, and electric charge. The continuity equation offers useful knowledge about the flow of fluids and their behavior as they go through a pipe or hose. The Continuity Equation is used on a variety of objects, including tubes, pipes, rivers, and ducts that carry gases or liquids. The continuity equation can be written in differential form, which is applied at a point, or in integral form, which is applied in a finite region. In this unit, equation of continuity and develop the equation of continuity in different type of co-ordinates are defined.

In analyzing the fluid motion, local and particle rates of change is very important. Also, acceleration of a fluid plays vital role in fluid dynamics. So, this chapter define such key factor of fluid dynamics. Appropriate boundary conditions are also developed.

3.2 *OBJECTIVES*

After completion of this unit learners will be able to:

(i) Define the concept of equation of continuity.

(ii) Develop the equation of continuity in various type of co-ordinates.

(iii) Describe the local and particle rates of changes

(iv) Define the acceleration of a fluid

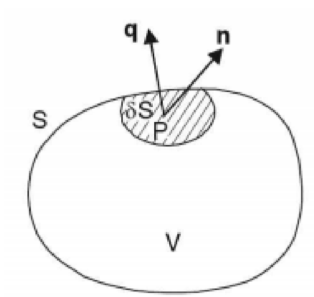
(v) Explain and derive the Boundary condition

3.3 *EQUATION OF CONTINUITY*

The law of conservation of mass states that fluid mass can be either created nor destroyed. The equation of continuity aims at expressing the law of conservation of mass in a mathematical form. Thus, in continuous motion, the equation of continuity expresses the fact that the increase in the mass of the fluid within any closed surface drawn in the fluid in any time must be equal to the excess of the mass that flows in over the mass that flows out.

3.3.1 EQUATION OF CONTINUITY BY EULER'S METHOD

Let S be an arbatrary small closed surface drawn in the compressible fluid enclosing a volume V and let S be taken fixed in space. Let $P(x, y, z)$ be any point of S and $\rho(x, y, z, t)$ be the fluid density at P at any time t. Let δS denote element of the surface S esclosing P. Let **n** be the unit outward-drawn normal at δS and let **q** be the fluid velocity at P. Then the normal component of **q** measured outwards from V is n. Thus



Rate of mass flow across $\delta S = \rho(\mathbf{n} \cdot \boldsymbol{q})\delta S$

∴ Total rate of mass flow across S

$$= \int_S \rho(\mathbf{n} \cdot \mathbf{q})\delta S = \int_V \nabla \cdot (\rho\mathbf{q}) dV$$

(By Gauss divergence theorem)

∴ Total rate of mass flow into $V = -\int_V \nabla \cdot (\rho \mathbf{q}) dV$ (3.1)

Again, the mass of the fluid within S at time $t = -\int_V \rho dV$

∴ Total rate of mass increase within $S = \frac{\partial}{\partial \mathrm{t}} \int_V \rho dV = \int_V \frac{\partial \rho}{\partial \mathrm{t}} dV$ (3.2)

Suppose that the region V of the fluid contains neither sources nor sinks (*i.e.*, there are no inlets or outlets through which fluid can enter or leave the region). Then by the law of conservation of the fluid mass, the rate of increase of the mass of fluid within V must be equal to the total rate of mass flowing into V. Hence from (3.1) and (3.2), we have

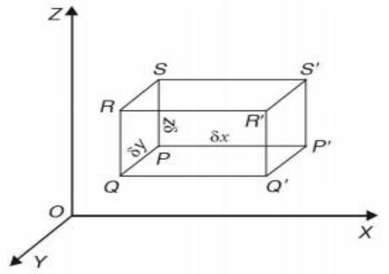
$$\int_V \frac{\partial \rho}{\partial t} dV = -\int_V \nabla \cdot (\rho \mathbf{q}) dV \qquad \text{or} \qquad \int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{q})\right] dV = 0$$

which holds for arbitrary small volumes V, if $\frac{\partial p}{\partial t} + \nabla \cdot (\rho q) = 0$ (3.3)

Equation (3.3) is called the equation of continuity, or the conservation of mass and it holds at all points of fluid free from sources and sinks.

3.3.2 THE EQUATION OF CONTINUITY IN CARTESIAN COORDINATES

Let there be a fluid particle at $P(x, y, z)$. Let $\rho(x, y, z, t)$ be the density of the fluid at P at any time t and let u, v, w be the velocity components at P parallel to the rectangular coordinate axes. Construct a small parallelepiped with edges $\delta x, \delta y, \delta z$ of lengths parallel to their respective coordinate axes, having P at one of the angular points as shown in figure. Then, we have



Mass of the fluid that passes in through the face $PQRS$

$$= (\rho\delta y\delta z)u \text{ per unit time } = f(x, y, z) \text{ say} \qquad (3.4)$$

$\therefore$ Mass of the fluid that passes out through the opposite face $P'Q'R'S$

$$= f(x + \delta x, y, z) \text{ per unit time } = f(x, y, z) + \delta x \frac{\partial}{\partial x} f(x, y, z) + \cdots \qquad (3.5)$$

(expanding by Taylor's theorem)

$\therefore$ The net gain in mass per unit time within the element (rectangular parallelepiped) due to flow through the faces PQRS and $P'Q'R'S'$ by using (3.4) and (3.5)

$=$ Mass that enters in through the face $PQRS$
$-$ Mass that leaves through the face $P'Q'R'S'$

$$= f(x, y, z) - \left[f(x, y, z) + \delta x \cdot \frac{\partial}{\partial x} f(x, y, z) + \cdots \right]$$

$$= -\delta x \cdot \frac{\partial}{\partial x} f(x, y, z), \text{ to the first order of approximation}$$

$$= -\delta x \cdot \frac{\partial}{\partial x} (\rho u \delta y \delta z), \text{ by (3.4)}$$

$$= -\delta x \delta y \delta z \frac{\partial(\rho u)}{\partial x} \qquad (3.6)$$

Similarly, the net gain in mass per unit time within the element due to flow through the faces $PP'S'S$ and $QQ'RR'$ $= -\delta x \delta y \delta z \frac{\partial(\rho v)}{\partial y}$ (3.7)

and the net gain in mass per unit time within the element due to flow through the faces $PP'QQ$ and $SS'R'R$ $= -\delta x \delta y \delta z \frac{\partial(\rho w)}{\partial z}$ (3.8)

∴ Total rate of mass flow into the elementary parallelepiped

$$= -\delta x\delta y\delta z\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right] \qquad (3.9)$$

Again, the mass of the fluid within the chosen element at time $t = \rho\delta x\delta y\delta z$

∴ Total rate of mass increase within the element

$$= \frac{\partial}{\partial t}(\rho\delta x\delta y\delta z) = \delta x\delta y\delta z\frac{\partial\rho}{\partial t} \qquad (3.10)$$

Suppose that the chosen region (bounded by the elementary parallelepiped) of the fluid contains neither sources nor sinks. Then by the law of conservation of the fluid mass, the rate of increase of the mass of the fluid within the element must be equal to the rate of mass flowing into the element. Hence from (6) and (7), we have

$$\delta x\delta y\delta z\frac{\partial\rho}{\partial t} = -\delta x\delta y\delta z\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right]$$

or $$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \qquad (3.11)$$

or $$\frac{\partial\rho}{\partial t} + \rho\frac{\partial u}{\partial x} + u\frac{\partial\rho}{\partial x} + \rho\frac{\partial v}{\partial y} + v\frac{\partial\rho}{\partial y} + \rho\frac{\partial w}{\partial z} + w\frac{\partial\rho}{\partial z} = 0$$

or $$\left[\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right]\rho + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

or $$\frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \qquad (3.12)$$

which is the desired equation of continuity in cartesian coordinates and it holds at all point of the fluid free from sources and sinks.

3.3.3 THE EQUATION OF CONTINUITY IN CYLINDRICAL COORDINATES

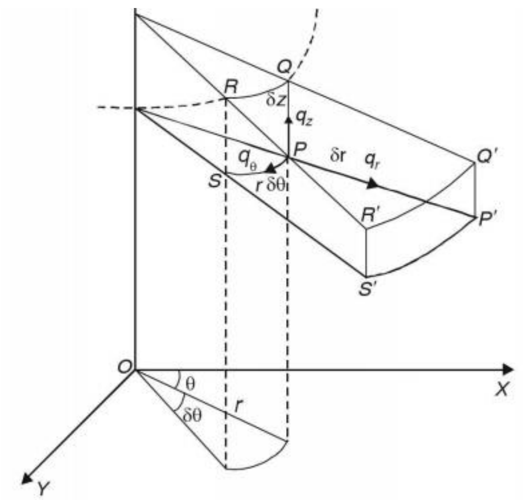
Let there be a fluid particle at P whose cylindrical coordinates are (r, θ, z), where $r \geq 0, 0 \leq \theta \leq 2\pi, -\infty < z < \infty$. Let $\rho(r, \theta, z, t)$ be the density of the fluid at P at any time t. With P as one corner construct a small curvilinear parallelepiped ($PQRS, P'Q'R'S'$) with its edges $SS' = \delta r$, arc $SP = r\delta\theta$ and $PQ = \delta z$. Let q_r, q_θ and q_z be the velocity components in the direction of the elements SS', arc SP and PQ respectively. Then, we have Mass of the fluid that passes in through the face PSRQ

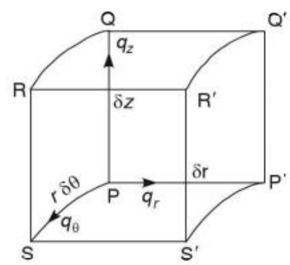
$= \rho \cdot r\delta\theta\delta z \cdot q_r$ per unit time $= f(r, \theta, z)$, say (3.13)

$\therefore$ Mass of the fluid that passes out through the opposite face $P'S'R'Q'$

$= f(r + \delta r, \theta, z)$ per unit time $= f(r, \theta, z) + \delta r \frac{\partial}{\partial r} f(r, \theta, z) + \cdots$ (3.14)

(expanding by Taylor's theorem)





$\therefore$ The net gain in mass per unit time within the chosen elementary parallelepiped (PQRS, $P'Q'R'S'$) due to flow through the faces $PSRQ$ and $P'SR'Q'$ by using (3.13) and (3.14)

$=$ Mass that enters in through the face $PQRS$

$-$ Mass that leaves through the face $P'Q'R'S'$

$$= f(r, \theta, z) - \left[f(r, \theta, z) + \delta r \cdot \frac{\partial}{\partial r} f(r, \theta, z) + \cdots \right]$$

$$= -\delta r \cdot \frac{\partial}{\partial r} f(r, \theta, z).$$ to the first order of approximation

$= -\delta r - \frac{\partial}{\partial r}(\rho r\delta\theta\delta z q_r)$, by (3.13)

$= -\delta r\delta\theta\delta z \frac{\partial(\rho r q_r)}{\partial r}$ (3.15)

Similarly, the net gain in mass per unit time within the element due to flow through the faces $SRR'S'$ and $QPP'Q'$

$= -\delta r\delta\theta\delta z \frac{\partial}{\partial\theta}(\rho q_\theta)$ (3.16) and the net gain in mass per unit time within the element due to flow through the faces PSSP and $QRR'Q'$

$$= -\delta r\delta\theta\delta z \frac{\partial}{\partial z}(\rho r q_z) = -r\delta r\delta\theta\delta z \frac{\partial(\rho q_z)}{\partial z} \qquad (3.17)$$

∴ Total rate of mass flow into the chosen element

$$= -\delta r\delta\theta\delta z\left[\frac{\partial}{\partial r}(\rho r q_r) + \frac{\partial}{\partial \theta}(\rho q_\theta) + r\frac{\partial}{\partial z}(\rho q_z)\right] \qquad (3.18)$$

Again, the mass of the fluid within the element at time $t = \rho r\delta r\delta\theta\delta z$

∴ Total rate of mass increase within the element

$$= \frac{\partial}{\partial \mathrm{t}}(\rho r\delta r\delta\theta\delta z) = r\delta r\delta\theta\delta z\frac{\partial\rho}{\partial t} \qquad (3.19)$$

Suppose that the chosen region of the element of the fluid contains neither sources nor sinks. Then by the law of conservation of the fluid mass, the rate of increase of the mass of the fluid within the element must be equal to the rate of mass flowing into the element. Hence from (3.18) and (3.19), we have

$$r\delta r\delta\theta\delta z\frac{\partial\rho}{\partial t} = -\delta r\delta\theta\delta z\left[\frac{\partial}{\partial r}(\rho r q_r) + \frac{\partial}{\partial \theta}(\rho q_\theta) + r\frac{\partial}{\partial z}(\rho q_z)\right]$$

or $$\frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r q_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho q_\theta) + \frac{1}{\partial z}(\rho q_z) = 0, \qquad (3.20)$$

which is the desired equation of continuity in cylindrical coordinates and it holds at all points of the fluid free from sources and sinks.

3.3.4 THE EQUATION OF CONTINUITY IN SPHERICAL POLAR COORDINATES

Let there be a fluid particle at P whose spherical polar coordinates are (r, θ, ϕ), where $r \geq 0, 0 \leq \phi \leq 2\pi, 0 \leq \theta \leq \pi$. Let $\rho(r, \theta, \phi, t)$ be the density of the fluid at P at any time t. With P as one corner construct a small curvilinear parallelopiped ($PQRS, P'Q'R'S'$) with its edges $PP' = \delta r$, are $PQ = r\delta\theta$, are $PS = r\sin\theta\delta\phi$. Let q_r, q_θ and q_ϕ be the velocity components in the direction of the elements PP', are PQ and arc PS respectively. Then, we have

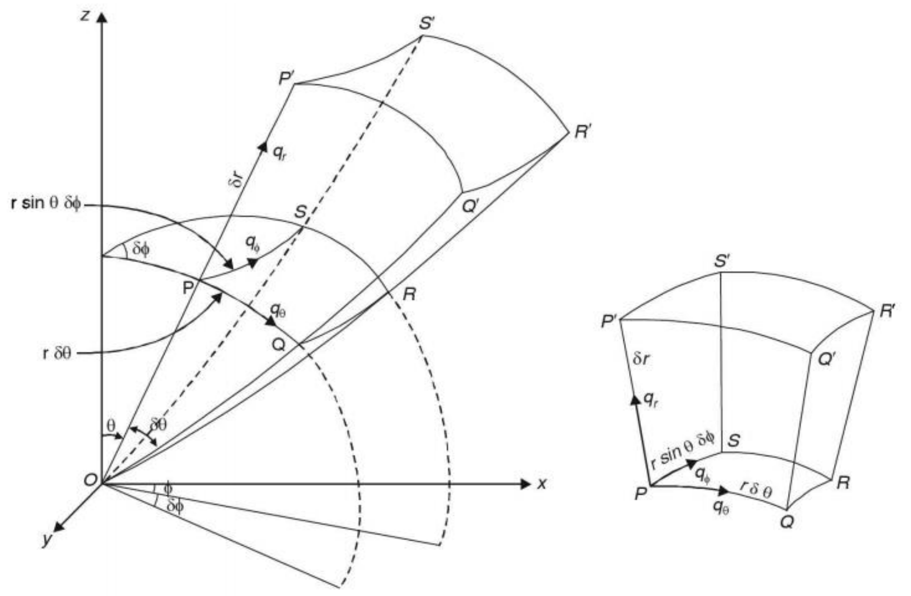
Mass of the fluid that passes in through the face $PQRS$

$$= \rho \cdot r\delta\theta\delta r\sin\theta\delta\phi \cdot q_r \text{ per unit time } = f(r, \theta, \phi), \text{ say} \qquad (3.21)$$

∴ Mass of the fluid that passes out through the opposite face $PQ'R'S$

$$= f(r + \delta r, \theta, \phi) = f(r, \theta, \phi) + \delta r \frac{\partial}{\partial r}(r, \theta, \phi) + \cdots \quad (3.22)$$

(expanding by Taylor's theorem)



∴ The net gain in mass per unit time within the chosen elementary parallelopiped (PQRS, $P'Q'R'S'$) due to flow through the faces $PQRS$ and $P'Q'R'S'$ by using (3.21)and (3.22)

= Mass that enters in through the face $PQRS$ - Mass that leaves through the face $P'Q'R'S'$

$$= f(r, \theta, \phi) - \left[f(r, \theta, \phi) + \delta r \cdot \frac{\partial}{\partial r}(r, \theta, \phi) + \cdots \right]$$

$= -\delta r \cdot \frac{\partial}{\partial r} f(r, \theta, \phi)$, to the first order of approximation

$$= -\delta r \cdot \frac{\partial}{\partial r}(\rho r^2 \sin\theta q, \delta\theta\delta\phi), \text{ by (3.21)} \quad (3.23)$$

Similarly, the net gain in mass per unit time within the element due to flow through the faces $PSS'P'$ and $QRR'Q'$

$$= -r\delta\theta \frac{\partial}{r\delta\theta}(\rho \cdot \delta r \cdot r\sin\theta\delta\phi \cdot q_\theta) \quad (3.24)$$

and the net gain in mass per unit time within the element due to flow through the faces $PQQ'P'$and $SRR'S'$

$$= -r\sin\theta\delta\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\rho\cdot\delta r r\delta\theta\cdot q_\theta) \qquad (3.25)$$

∴ Total rate of mass flow into the elementary parallelepiped

$$= -\delta r\delta\theta\delta z\left[\sin\theta\frac{\partial}{\partial r}(\rho r^2 q_r) + r\frac{\partial}{\partial\theta}(\rho\sin\theta q_\theta) + r\frac{\partial}{\partial\phi}(\rho q_\phi)\right] \qquad (3.26)$$

Again, the mass of the fluid within the chosen element at time $t = -\rho\delta r.r\delta\theta.r\sin\theta\delta\phi$

∴ Total rate of mass increase within the element

$$= \frac{\partial}{\partial t}(\rho r^2\sin\theta\delta r\delta\theta\delta\phi) = r^2\sin\theta\delta r\delta\theta\delta\phi\frac{\partial\rho}{\partial t} \qquad (3.27)$$

Suppose that the chosen region of the fluid contains neither sources nor sinks. Then by the law of conservation of the fluid mass, the rate of increase of the fluid within the element must be equal to the rate of mass flowing into the element. Hence from (3.26) and (3.27), we have

$$-\delta r\delta\theta\delta z\left[\sin\theta\frac{\partial}{\partial r}(\rho r^2 q_r) + r\frac{\partial}{\partial\theta}(\rho\sin\theta q_\theta) + r\frac{\partial}{\partial\phi}(\rho q_\phi)\right] = r^2\sin\theta\delta r\delta\theta\delta\phi\frac{\partial\rho}{\partial r}$$

or

$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 q_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho\sin\theta q_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\rho q_\phi) = 0, \qquad (3.28)$$

which is the desired equation of continuity in spherical polar coordinates and it bolds at all points of the fluid free from sources and sinks.

3.4 *LOCAL AND PARTICLES RATE OF CHANGES*

Suppose a particle of fluid moves form $P(x, y, z)$ at time t to $P'(x + \delta x, y + \delta y, z + \delta z)$ at time $t + \delta$ t. Let $f(x, y, z, t)$ be a scalar function associated with some properties of fluid. Then, the motion of the particle from P to P ' the total change of is

$$\delta f = \frac{\delta f}{\delta x}\delta x + \frac{\delta f}{\delta y}\delta y + \frac{\delta f}{\delta z}\delta z + \frac{\delta f}{\delta t}\delta t$$

Thus, the total rate of change of f at a point P at a time t.
In the motion of the particle,

$$\frac{df}{dt} = \lim_{\delta t \to 0}\left(\frac{\delta f}{\delta x}\right)$$
$$= \frac{\delta f}{\delta x}\frac{dx}{dt} + \frac{\delta f}{\delta y}\frac{dy}{dt} + \frac{\delta f}{\delta z}\frac{dz}{dt} + \frac{\delta f}{\delta t}$$
$$= u\frac{\delta f}{\delta x} + v\frac{\delta f}{\delta y} + w\frac{\delta f}{\delta z} + \frac{\delta f}{\delta t}$$

If $\mathrm{q} = [\mathrm{u}, \mathrm{v}, \mathrm{w}]$ is the velocity of the fluid particle at P

$$\frac{\mathrm{df}}{\mathrm{dt}} = \mathrm{q}.\nabla\mathrm{f} + \frac{\partial \mathrm{f}}{\partial \mathrm{t}} \qquad (3.29)$$

Similarly, for a velocity function $\mathrm{F}(\mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{t})$ associtated with some property of a fluid.

$$\frac{\mathrm{dF}}{\mathrm{dt}} = \mathrm{q}.\nabla\mathrm{F} + \frac{\partial \mathrm{F}}{\partial \mathrm{t}} \qquad (3.30)$$

Hence, both the scalar and vector function of position and time, By operation equality $\frac{\mathrm{d}}{\mathrm{dt}} = \mathrm{q} \cdot \nabla + \frac{\partial}{\partial \mathrm{t}}$, provided that those functions are associated with the properties of the moving fluid.

In the obtaining equation (3.29) and (3.30), we considered total change. When the fluid particle moves from $p(x, y, z)$ to $P'(x + \delta x, y + \delta y, z + \delta z)$ in time δt.

Thus, $\frac{\mathrm{df}}{\mathrm{dt}}, \frac{\mathrm{dF}}{\mathrm{dt}}$ are a total differentiation following the fluid particles are called particle rates of change.

On the other hand, particle time derivative $\frac{\partial \mathrm{f}}{\partial \mathrm{t}}, \frac{\partial \mathrm{F}}{\partial \mathrm{t}}$ are only the time rates of change at the point $p(x, y, z)$.

Consider fixed in space at a point $\mathrm{p}(\mathrm{x}, \mathrm{y}, \mathrm{z})$ they are the local rates of change. It follows that $\mathrm{q}.\nabla\mathrm{f}$ or $\mathrm{q}.\nabla\mathrm{F}$.

3.5 *ACCELERATION OF A FLUID*

Let velocity is a vector function of position and time and thus has three components *u*, *v*, and *w*, each a scalar field in itself:

$$\mathbf{V}(\mathbf{r}, t) = \mathrm{i}u(x, y, z, t) + \mathrm{j}v(x, y, z, t) + \mathbf{k}w(x, y, z, t) \qquad (3.31)$$

To write Newton's second law for an infinitesimal fluid system, we need to calculate the acceleration vector field a of the flow. Thus, we compute the total time derivative of the velocity vector:

$$\mathrm{a} = \frac{d\mathbf{V}}{dt} = \mathrm{i}\frac{du}{dt} + \mathrm{j}\frac{dv}{dt} + \mathbf{k}\frac{dw}{dt} \qquad (3.32)$$

Since each scalar component (u, v, w) is a function of the four variables (x, y, z, t), we use the chain rule to obtain each scalar time derivative. For example,

$$\frac{du(x, y, z, t)}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

But, by definition, dx/dt is the local velocity component u, and $dy/dt = v$, and $dz/dt = w$. The total time derivative of u may thus be written as follows, with exactly similar expressions for the time derivatives of v and :

$$\begin{aligned} a_x &= \frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{\partial u}{\partial t} + (\mathbf{V}\cdot\nabla)u \\ a_y &= \frac{dv}{dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \frac{\partial v}{\partial t} + (\mathbf{V}\cdot\nabla)v \qquad (3.33) \\ a_z &= \frac{dw}{dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = \frac{\partial w}{\partial t} + (\mathbf{V}\cdot\nabla)w \end{aligned}$$

Summing these into a vector, we obtain the total acceleration:

$$\mathrm{a} = \frac{dV}{dt} = \frac{\partial V}{\partial t} + \left(u\frac{\partial V}{\partial x} + v\frac{\partial V}{\partial y} + w\frac{\partial V}{\partial z}\right) = \frac{\partial w}{\partial t} + (\mathbf{V}\cdot\nabla)V \qquad (3.34)$$

The term $\partial\mathbf{V}/\partial t$ is called the local acceleration, which vanishes if the flow is steadythat is, independent of time. The three terms in parentheses are called the convective acceleration, which arises when the particle moves through regions of spatially varying velocity, as in a nozzle or diffuser. Flows that are nominally "steady" may have large accelerations due to the convective terms.

3.6 *BOUNDARY CONDITION*

When fluid is in contact with a rigid solid surface (or with another unmixed fluid), the following boundary condition must be satisfied in order to maintain contact:

The fluid and the surface with which contact is preserved must have the same velocity normal to the surface.

Let $\mathbf{n}$ denote a normal unit vector drawn at the point P of the surface of contact and let $\mathbf{q}$ denote the fluid velocity at P. When the rigid surface of contact is at rest, we must have $\mathbf{q} \cdot \mathbf{n} = 0$ at each point of the surface. This expresses the condition that the normal velocities are both zero and hence the fluid velocity is tangential to the surface at its each point as shown in Fig. (i).

Next, let the rigid surface be in motion and let $\mathbf{u}$ be its velocity at P (refer Fig (ii)]. Then we must have

$$\mathbf{q} \cdot \mathbf{n} = \mathbf{u} \cdot \mathbf{n} \ \text{ or } \ (\mathbf{q} - \mathbf{u}) \cdot \mathbf{n} = 0,$$

which expresses the fact that there must be no normal velocity at P between boundary and fluid, that is, the velocity of the fluid relative to the boundary is tangential to the boundary at its each point.

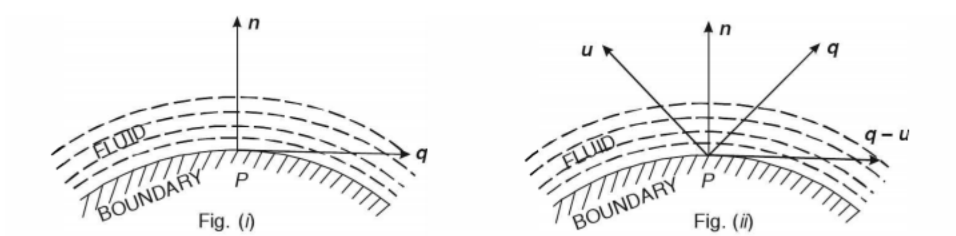


Fig. (*i*) Fig. (*ii*)

Remark. For inviscid fluid the above condition must be satisfied at the boundary. However, for viscous fluid (in which there is no slip), the fluid and the surface with which contact is maintained must also have the same tangential velocity at P.

The pressure of the fluid must act normal to the boundary.

Again, let S denote the surface of separation of two fluids (which do not mix). Then the following additional condition must be satisfied:

The pressure must be continuous at the boundary as we pass from one side of S to the other.

3.6.1 CONDITIONS AT A BOUNDARY SURFACE

We propose to derive the differential equation satisfied by a boundary surface of a fluid. Thus, we discuss the following problem:

To find the condition that the surface $F(\boldsymbol{r}, t) = 0$ or $F(x, y, z, t) = 0$ may be a boundary surface. For figure, refer figure (ii) of Sec. 3.5. Let P be a point on the moving boundary surface

$$F(\mathbf{r}, t) = 0. \qquad (3.35)$$

where the fluid velocity is $\mathbf{q}$ and the velocity of the surface is $\mathbf{u}$.

Now in order to preserve contact, the fluid and the surface with which contact is to be maintained must have the same velocity normal to the surface. Thus, we have

$$\mathbf{q} \cdot \mathbf{n} = \mathbf{u} \cdot \mathbf{n} \qquad \text{or} \qquad (\mathbf{q} - \mathbf{u}) - \mathbf{n} = 0 \qquad (3.36)$$

where $\mathbf{n}$ in the unit normal vector drawn at P on the boundary surface (3.35). We know that the direction ratios of $\mathbf{n}$ are $\partial F/\partial x, \partial F/\partial y, \partial F/\partial z$. Again,

$$\nabla F = (\partial F/\partial x)\mathbf{i} + (\partial F/\partial y)\mathbf{j} + (\partial F/\partial z)\mathbf{k}, \qquad (3.37)$$

which shows that $\mathbf{n}$ and ∇F are parallel vectors and hence we may write $\mathbf{n} = k\nabla F$. With this value of $\mathbf{n}$, (3.37) reduces to

$$(\mathbf{q} - \mathbf{u}) \cdot k\nabla F = 0 \qquad \text{so that} \qquad \mathbf{q} \cdot \nabla F = \mathbf{u} \cdot \nabla F \qquad (3.38)$$

Let $P(\mathbf{r}, t)$ move to a point $Q(\mathbf{r} + \delta\mathbf{r}, t + \delta t)$ in time δt. Then Q must satisfy the equation of the boundary surface (3.35), at time $t + \delta t$, namely

$$F(\mathbf{r} + \delta\mathbf{r}, t + \delta t) = 0$$

Expanding by Taylor's theorem, the above equation gives

$F(\mathbf{r},t)+\delta\mathbf{r}\cdot\nabla\mathbf{F}+\delta t\left(\frac{\partial F}{\partial t}\right)=0$ or $\frac{\partial F}{\partial t}+\frac{\delta\mathbf{r}}{\delta t}\cdot\nabla\mathbf{F}=0$, using (3.35) (3.39)

Proceeding to the limits as $\delta\mathbf{r}\to 0, \delta t\to 0$ and noting that

$\lim_{\partial t\to 0}\frac{\delta\mathbf{r}}{\delta t}=\frac{d\mathbf{r}}{dt}=\mathbf{u}$, (3.39) gives $\partial F/\partial t+\mathbf{u}\cdot\nabla\mathbf{F}=0$ (3.40)

or $\partial F/\partial t+\mathbf{q}\cdot\nabla\mathbf{F}=0$, using (3.38) (3.41)

which is the required condition for $F(r,t)$ to be a boundary surface.

Remark 1. Let $\mathbf{q}=u\mathbf{i}+v\mathbf{j}+w\mathbf{k}$. Then (3.41) may be re-written as

$$\frac{\partial F}{\partial t}+(u\mathbf{i}+v\mathbf{j}+w\mathbf{k})\cdot\left(\frac{\partial F}{\partial x}\mathbf{i}+\frac{\partial F}{\partial y}\mathbf{j}+\frac{\partial F}{\partial z}\mathbf{k}\right)=0$$

or $\frac{\partial F}{\partial t}+u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}+w\frac{\partial F}{\partial z}=0$ or $\frac{DF}{Dt}=0$ (3.42)

where $D=\partial/\partial t+u(\partial/\partial x)+v(\partial/\partial y)+w(\partial/\partial zz)$

(3.42) presents the required condition in cartesian coordinates for $F(x,y,z,t)=0$ to be a boundary surface.

Remark 2. The normal velocity of the boundary

$$=\mathbf{u}\cdot\mathbf{n}=\mathbf{u}\cdot\frac{\nabla\mathbf{F}}{|\nabla F|}$$

$$=\frac{-(\partial F/\partial t)}{|(\partial F/\partial x)\mathbf{i}+(\partial F/\partial y)\mathbf{j}+(\partial F)\mathbf{k}|}, \text{ by (3.37) and (3.40)}$$

$$=\frac{-(\partial F/\partial t)}{\sqrt{\{(\partial F/\partial x)^2+(\partial F/\partial y)^2+(\partial F/\partial z)^2\}}} \quad (3.43)$$

$$=\frac{u(\partial F/\partial x)+v(\partial F/\partial y)+w(\partial F/\partial z)}{\sqrt{\{(\partial F/\partial x)^2+(\partial F/\partial y)^2+(\partial F/\partial z)^2\}}}, \text{ using (3.42)} \quad (3.44)$$

3.6.2 EXAMPLES BASED ON EQUATION OF CONTINUITY

Example 1. The particles of a fluid move symmetrically in space with regard to a fixed center; prove that the equation of continuity is

$$\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial r}+\frac{\rho}{r^2}\frac{\partial}{\partial r}(r^2u)=0,$$ where u is the velocity at distance r :

Solution. Here we have spherical symmetry.

$$\frac{\partial\rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho u)=0 \text{ or } \frac{\partial\rho}{\partial t}+\frac{1}{r^2}\left\{r^2u\frac{\partial\rho}{\partial r}+\rho\frac{\partial}{\partial r}(r^2\rho u)\right\}=0$$

or $\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial r}+\frac{\rho}{r^2}\frac{\partial}{\partial r}(r^2\rho u)=0.$

Example 2. A mass of fluid moves in such a way that each particle describes a circle in one plane about a fixed axis; show that the equation of continuity is $\partial\rho/\partial t+\partial(\rho\omega)/\partial\theta=0,$ where ω is the angular velocity of a particle whose azimuthal angle is θ at time t

Solution. Here the motion is confined in a plane. Consider a fluid particle P, whose polar coordinates are (r,θ). Let P describe a circle of radius r. With P as one comer, consider an element $PQRS$ such that $PS=\delta r$ and arc $PQ=r\delta\theta$. Here there is no motion of the fluid along PS. The rate of the excess of the flow-in over the flow-out along PQ

$$=-r\delta\theta\frac{\partial}{r\,\partial\theta}(\rho\cdot r\omega-\delta r)$$

Again, the total mass of the fluid within the element $=\rho\cdot\delta r\cdot r\delta\theta.$

The rate of increase in mass of the element $=\frac{\partial}{\partial t}(\rho r\delta r 8\theta)$

Hence the equation of continuity is given by

$$\frac{\partial}{\partial r}(\rho r\delta r\delta\theta)=-r\delta\theta\frac{\partial}{r\,\partial\theta}(\rho r\omega\delta r) \quad \text{or} \quad r\delta r\delta\theta\frac{\partial\rho}{\partial r}=-r\delta r\delta\theta\frac{\partial}{\partial\theta}(\rho\omega)$$

or $$\partial\rho/\partial t+\partial(\rho\omega)/\partial\theta=0,$$

Example 3. Show that the surface $\frac{x^2}{a^2k^2t^4}+kt^2\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)=1$ is a possible form of boundary surface of a liquid at time t.

Solution. The given surface

$$F(x,y,z,t)=\frac{x^2}{a^2k^2t^4}+kt^2\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)-1=0 \qquad (3.45)$$

an be a possible boundary surface of a liquid, if it satisfies the boundary condition

$$\partial F/\partial t+u(\partial F/\partial x)+v(\partial F/\partial y)+w(\partial F/\partial z)=0 \qquad (3.46)$$

the same values of u,v,w satisfy the equation of continuity

$$\partial u/\partial x+\partial v/\partial y+\partial w/\partial z=0 \qquad (3.47)$$

From (3.45)

$$\frac{\partial F}{\partial t}=-\frac{4x^2}{a^2k^2t^5}+2kt\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right),\frac{\partial F}{\partial x}=\frac{2x}{a^2k^2t^4},\frac{\partial F}{\partial y}=\frac{2kt^2y}{b^2},\frac{\partial F}{\partial z}=\frac{2kt^2z}{c^2}$$

With these values, (3.46) reduces to

$$-\frac{4x^2}{a^2k^2t^5}+2kt\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)+\frac{2xu}{a^2k^2t^4}+\frac{2kt^2yv}{b^2}+\frac{2kt^2zw}{c^2}=0,$$
$$\frac{2x}{a^2k^2t^4}\left(u-\frac{2x}{t}\right)+\frac{2kyt}{b^2}(y+vt)+\frac{2ktz}{c^2}(z+wt)=0,$$

which is identically satisfied if we take

$$u=2x/t,\qquad v=-y/t\qquad w=-z/t \tag{3.48}$$

From (3.48) $\quad \frac{\partial u}{\partial x}=\frac{2}{t}\qquad \frac{\partial v}{\partial y}=-\frac{1}{t}\qquad \frac{\partial w}{\partial z}=-\frac{1}{t}$ (3.49)

Using (3.49), we find that (3.47) is also satisfied by the above values of u, v and w. Hence (3.45) is possible boundary surface with velocity components given by (3.48).

Example 4. If the velocity distribution is $\mathbf{q}=\mathbf{i}Ax^2y+\mathbf{j}By^2zt+\mathbf{k}Czt^2$, where A, B, C, are constants, then find the the acceleration and velocity components.

Solution. The acceleration $\boldsymbol{a}=a_x\mathbf{i}+a_y\mathbf{j}+a_z\mathbf{k}$ is given by

$$\boldsymbol{a}=\frac{\partial \mathbf{q}}{\partial t}+u\frac{\partial \mathbf{q}}{\partial x}+v\frac{\partial \mathbf{q}}{\partial y}+w\frac{\partial \mathbf{q}}{\partial z} \tag{3.50}$$

Also $\quad \mathbf{q}=u\mathbf{i}+v\mathbf{j}+w\mathbf{k}=\mathbf{i}Ax^2y+\mathbf{j}By^2zt+\mathbf{k}Czt^2$ (3.51)

Hence, $\quad u=Ax^2y,\ v=By^2zt,\ w=Czt^2$ (3.52)

Using (3.51) and (3.52), (3.50) reduces to

$$\boldsymbol{a}=By^2z\mathbf{j}+2Czt\mathbf{k}+Ax^2y\times(2\,Axy\mathbf{i})+By^2zt\left(Ax^2\mathbf{i}+2Byzt\mathbf{j}\right)$$
$$+Czt^2\left(By^2t\mathbf{j}+Ct^2\mathbf{k}\right)$$
$$=A(2Ax^3y^2+Bx^2y^2zt)\mathbf{i}+B(y^2z+2By^3z^2t^2+Cy^2zt^3)\mathbf{j}+C(2zt+Czt^4)\mathbf{k}$$

The components of the acceleration (a_x, a_y, a_z) are given by

$$a_x=A(2Ax^3y^2+Bx^2y^2zt),\ a_y=B(y^2z+2By^3z^2t^2+Cy^2zt^3),\ a_z=C(2zt+Czt^4)$$

Example 5. Determine the acceleration at the point (2,1,3) at $t=0.5$ sec, if $u=yz+t$, $v=xz-t$ and $w=xy$.

Solution. Velocity field $\mathbf{q}$ at the point (x,y,z) is given by

$$\mathbf{q}=\boldsymbol{u}\mathbf{i}+v\mathbf{j}+w\mathbf{k}=(yz+t)\mathbf{i}+(xz-t)\mathbf{j}+xy\mathbf{k}. \tag{3.52}$$

The acceleration $\boldsymbol{a}=a_x\mathbf{i}+a_y\mathbf{j}+a_z\mathbf{k}$ is given by

$\boldsymbol{a} = \partial \boldsymbol{q}/\partial t + u(\partial \boldsymbol{q}/\partial t) + v(\partial \boldsymbol{q}/\partial t) + w(\partial \boldsymbol{q}/\partial t)$

$= (\mathbf{i} - \mathbf{j}) + (yz + t)(z\mathbf{j} + y\mathbf{k}) + (xz - t)(z\mathbf{i} + x\mathbf{k}) + xy(y\mathbf{i} + x\mathbf{j})$

$= (1 + xz^2 + xy^2 - tz)\mathbf{i} + (-1 + yz^2 + x^2z + zt)\mathbf{j} + (y^2z + x^2z + yt - xt)\mathbf{k}$

∴ Acceleration at $(2,1,3)$ at $t = 0.5$ is given by $\boldsymbol{a} = 19.5\mathbf{i} + 13.5\mathbf{j} + 6.5\mathbf{k}$

Hence the components a_x, a_y, a_z of acceleration are given by

$a_x = 19.5$, $a_y = 13.5$ and $a_z = 6.5$

3.7 *SUMMARY*

This unit explains the following topics:

(i) Description of equation of continuity

(ii) Derivation of the equation of continuity in different coordinates

(iii) Definition of local and particle rates of change

(iv) Concept of acceleration of a fluid

(v) Explanation of Boundary condition

3.8 *GLOSSARY*

(i) Fluid

(ii) Velocity

(iii) Equation of continuity

(iv) Acceleration

(v) Rigid Boundary

3.9 *REFERENCES*

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3.10 *TERMINAL QUESTIONS*

1. What is a local and particle rates of changes?

2. Define the equation of continuity?

3. Derive the equation of continuity in spherical polar coordinates.

4. Derive the equation of continuity in cylindrical coordinates.

5. What is boundary condition? Derive the condition at boundary surface.

6. Given the eulerian velocity vector field

$$V = \mathbf{i}3\mathrm{t} + \mathbf{j}xz + \mathbf{k}ty^2$$

find the total acceleration of a particle.

Sol. $\frac{\mathrm{dV}}{\mathrm{dt}} = \mathbf{i}3 + \mathbf{j}(3tz + txy^2) + \mathbf{k}(y^2 + 2txyz)$

7. Determine the acceleration of a fluid particle from the following flow field:

$$\mathbf{q} = \mathbf{i}(Axy^2t) + \mathbf{j}(Bx^2yt) + \mathbf{k}(Cxyz).$$

Sol. $a_x = A(xy^2 + Axy^4t + 2Bx^3y^2t^2),$

$a_y = B(x^2y + 2Ax^2y^3t + Bx^4yt^2),$

$a_z = C(Axy^3z + Bx^3yzt + x^2y^2z)$

Course Name: FLUID MECHANICS

Course Code: MAT604

BLOCK-II

EQUATION OF MOTION OF A FLUID

UNIT 4: *PRESSURE AT A POINT IN A FLUID AT REST OR IN A MOTION*

CONTENTS:

4.1 INTRODUCTION

In this chapter, we will examine a significant category of problems where the fluid is either stationary or moving without any relative motion between adjacent particles. In both cases, there will be no shearing stresses in the fluid, and the only forces acting on the surfaces of the particles will be due to pressure. Therefore, our main focus is on studying pressure and its variation throughout a fluid. The absence of shearing stresses simplifies the analysis considerably, enabling us to achieve relatively straightforward solutions to many important practical problems.

4.2 OBJECTIVES

Upon reading this unit learner will be able to:

(i) Understand how pressure varies in a stationary fluid.

(ii) Evaluate the movement of fluids in containers subjected to linear acceleration or rotation.

4.3 FLUID PRESSURE AT A POINT

Imagine a small area dA within a large mass of fluid. When the fluid is at rest, the force exerted by the surrounding fluid on this area dA will always be perpendicular to the surface of dA. Pressure is the normal force that a fluid applies per unit area, denoted as p. Thus, mathematically, the pressure at a point in a stationary fluid is defined as

$$p = \frac{dF}{dA}$$

If the force (F) is evenly distributed across the area (A), the pressure at any point is defined as

$$p = \frac{F}{A} = \frac{Force}{Area}$$

Therefore, force or pressure force $F = p \times A$.

Pressure is measured in various units, including:

(i) In the MKS system: *kgf/m²* and *kgf/cm²*.

(ii) In the SI system: Newton/m² (*N/m²*) and Newton/mm² (*N/mm²*). *N/m²* is also referred to as Pascal (*Pa*).

Other frequently used units of pressure include:

$$kPa = kilo\ pascal = 1000\ K/m^2.$$

$$bar = 100\ kPa = 10^5 K/m^2.$$

4.4 PASCAL'S LAW

Pascal's law states that the pressure or intensity of pressure at a point in a static fluid is the same in all directions. This can be demonstrated as follows:

Consider a small wedge-shaped fluid element within a stationary fluid mass, as shown in Figure 4.1. Let the width of the element perpendicular to the plane of the paper be unity, with $p_x, p_y,$ and p_z representing the pressure or intensity of pressure acting on the faces AB, AC, and BC, respectively. Let $\angle ABC = \theta$. The forces acting on the element include:

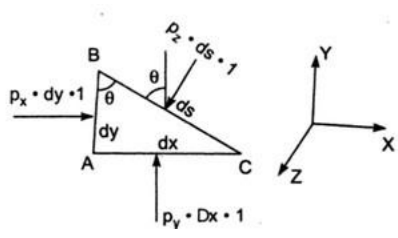


***Figure 4.1: Forces on a fluid element*.**

(i) Pressure forces normal to the surfaces.

(ii) Weight of element acting in the vertical direction.

The forces acting on the faces are:

Force on the face $AB = p_x \times$ Area of face AB

$$= p_x \times dy \times 1.$$

In a similar manner, the force on the face $AC = p_y \times dx \times 1.$

In a similar manner, the force on the face $BC = p_z \times ds \times 1.$

Weight of element = (Mass of element)$\times g$

$$= (\text{Volume} \times \rho) \times g = \left(\frac{AB \times AC}{2} \times 1\right) \times \rho \times g,$$

where ρ is the density of fluid.

Resolving the forces in x-direction, we obtain

$$(p_x \times dy \times 1) - (p_z \times ds \times 1) sin(90° - \theta) = 0$$

or

$$(p_x \times dy \times 1) - (p_z \times ds \times 1) cos\, \theta = 0.$$

But from Figure 4.1, $ds\, cos\, \theta = AB = dy$, therefore

$$(p_x \times dy \times 1) - p_z\, dy = 0,$$

or

$$p_x = p_z. \quad (4.1)$$

In a similar manner, resolving the forces in y-direction, we obtain

$$(p_y \times dx \times 1) - (p_z \times ds \times 1) cos(90° - \theta) - \left(\frac{dx \times dy}{2} \times 1\right) \times \rho \times g = 0$$

or

$$(p_y \times dx) - (p_z \times ds \times sin\, \theta) - \left(\frac{dxdy}{2} \times \rho \times g\right) = 0.$$

But $ds\, sin\, \theta = dx$. Since the element is very small, its weight can be considered negligible, so

$$p_y dx - p_z dx = 0,$$

i.e. $$p_y = p_z. \quad (4.2)$$

From equation (4.1) and (4.2), we get

$$p_x = p_y = p_z.$$

This demonstrates that the pressure at any point is equal in the x, y, and z directions. Because the selection of the fluid element was arbitrary, it implies that the pressure at any point is uniform in all directions.

4.5 PRESSURE VARIATION IN A FLUID AT REST

The pressure at any point in a stationary fluid is determined by the hydrostatic law, which states that the rate of pressure increase in the downward vertical direction is equal to the specific weight of the fluid at that point. This can be demonstrated as follows:

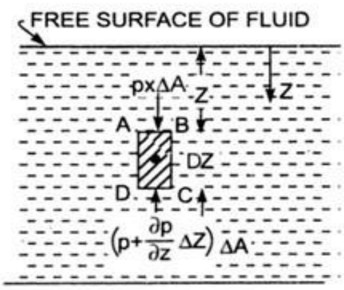


Figure 4.2: Forces on a fluid element.

Let's examine a small fluid element given in Figure 4.2.

Let $\Delta A =$ Cross-sectional area of element

$\Delta Z =$ Height of fluid element

$p =$ Pressure on face AB

$Z =$ Distance of fluid element from free surface.

The forces acting on the fluid element are as follows:

1. The pressure force on face AB is $p \times \Delta A$, acting downward, perpendicular to face AB.
2. The pressure force on face CD is $\left(p + \frac{\partial p}{\partial z}\Delta Z\right) \times \Delta A$, acting upward, perpendicular to face CD.
3. The weight of fluid element is $\rho \times g \times (\Delta A \times \Delta Z)$.

4. The pressure forces on surfaces BC and AD are equal and opposite. For the equilibrium of the fluid element, we have

$$p\Delta A - \left(p + \frac{\partial p}{\partial Z}\Delta Z\right)\Delta A + \rho \times g \times (\Delta A \times \Delta Z) = 0$$

or

$$p\Delta A - p\Delta A - \frac{\partial p}{\partial Z}\Delta Z\Delta A + \rho \times g \times \Delta A \times Z = 0$$

or

$$-\frac{\partial p}{\partial Z}\Delta Z\Delta A + \rho \times g \times \Delta A\Delta Z = 0$$

or

$$\frac{\partial p}{\partial z}\Delta Z\Delta A = \rho \times g \times \Delta A\Delta Z \text{ or } \frac{\partial p}{\partial z} = \rho \times g$$

Therefore, $\frac{\partial p}{\partial z} = \rho \times g = w$ [Since, $\rho \times g = w$] (4.3)

where w is the weight density of fluid.

Equation (4.3) indicates that the rate of pressure increase vertically equals the density of the fluid multiplied by gravitational acceleration. This principle is known as the **Hydrostatic Law**.

By integrating equation (4.3) for liquids, we obtain

$$\int dp = \int \rho g Z$$

$$p = \rho g Z. \quad (4.4)$$

where p represents the pressure relative to atmospheric pressure, and Z denotes the height of the point from free surfaces.

From equation (4.4), we obtain $Z = \frac{p}{\rho \times g}$ (4.5)

Here Z is referred to as the **Pressure Head**.

Problem 4.5.1 *A hydraulic press has a ram with a diameter of* $20\ cm$ *and a plunger with a diameter of* $3.5\ cm$*. Determine the weight lifted by the hydraulic press when a force of 500 N is applied to the plunger.*

Solution. Diameter of ram, $D = 20cm = 0.2m$

Diameter of plunger, $d = 3.5cm = 0.035m$

Force on plunger, $F = 500$ N

We have to find weight lifted, i.e., W.

Area of ram,

$A = \frac{\pi}{4}D^2 = \frac{\pi}{4}(0.2)^2 = 0.03142 \text{ m}^2$

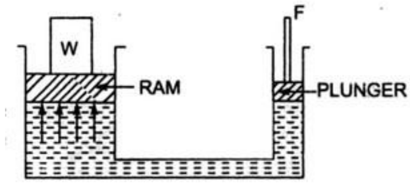


Figure 4.3

Area of plunger.

$$a = \frac{\pi}{4}d^2 = \frac{\pi}{4}(0.035)^2 = .00096 \text{ m}^2.$$

Pressure intensity due to plunger

$$= \frac{\text{Force on plunger}}{\text{Area of plunger}} = \frac{F}{a} = \frac{500}{.00096} \text{ N/m}^2.$$

According to Pascal's law, the pressure intensity will be uniformly transmitted in all directions. Therefore, the pressure intensity at the ram

$$= \frac{500}{.00096} = 520833.33 \ N/\text{m}^2.$$

$Also, the\ pressure\ intensity\ at\ ram = \frac{\text{Weight}}{\text{Area of ram}} = \frac{W}{A} = \frac{W}{.03142}$ N/ m^2,

$$\text{i.e.,} \quad \frac{W}{.03142} = 520833.33$$

$$\therefore \text{ Weight } = 520833.33 \times .03142 = 16364.58 \text{ N}$$

$$= \mathbf{16.36458\ kN}.$$

Problem 4.5.2 A hydraulic press has a ram with a diameter of 30 cm and a plunger with a diameter of 2 cm. It is used to lift a weight of 30kN. Determine the force required at the plunger.

Solution. Diameter of ram, $D = 30cm = 0.3m$

$\therefore$ Area of ram, $A = \frac{\pi}{4}D^2 = \frac{\pi}{4}(0.3)^2 = 0.07068\ m^2$

Diameter of plunger, $d = 2cm = 0.02m$

$\therefore$ Area of plunger, $a = \frac{\pi}{4}d^2 = \frac{\pi}{4}(0.02)^2 = 3.142 \times 10^{-4} m^2$

Weight lifted, $W = 30kN = 30 \times 1000\ N = 30000\ N.$

Pressure intensity developed due to plunger $= \frac{\text{Force}}{\text{Area}} = \frac{F}{a}.$

According to Pascal's law, the pressure intensity will be uniformly transmitted in all directions. Therefore, the pressure transmitted at the ram $= \frac{F}{a}$

$$Force\ acting\ on\ ram = \text{Pressure intensity} \times \text{Area of ram}$$

$$= \frac{F}{a} \times A = \frac{F \times .07068}{3.142 \times 10^{-4}} \text{ N}$$

$$\text{But force acting on ram} = \text{Weight lifted} = 30000 \text{ N}$$

$$30000 = \frac{F \times .07068}{3.142 \times 10^{-4}}$$

$$F = \frac{30000 \times 3.142 \times 10^{-4}}{.07068} = \mathbf{133.36\ N.}$$

Problem 4.5.3 Calculate the pressure exerted by a liquid column with a height of 0.4 m for the following liquids:

(a) Water, (b) Oil with a specific gravity of 0.9, (c) Mercury with a specific gravity of 13.6.

Assume the density of water, $\rho = 1000\ kg/\text{m}^3$.

Solution. Height of liquid column,

$$Z = 0.4 \text{ m.}$$

The pressure at any point in a liquid is determined by equation (4.5) as follows

$$p = \rho g Z$$

$$\rho = 1000 \text{ kg/m}^3$$

$$p = \rho g Z = 1000 \times 9.81 \times 0.4 = 3924 \text{ N/m}^2$$

$$= \frac{3924}{10^4} \text{ N/cm}^2 = \mathbf{0.3924} \text{ N/cm}^2.$$

(b) For oil of specific gravity 0.9 ,

We know that the density of a fluid is the product of its specific gravity and the density of water.

∴ Density of oil,

$$\rho_0 = \text{Sp. gr. of oil} \times \text{Density of water}$$

$$= 0.9 \times \rho = 0.9 \times 1000 = 900\ kg/\text{m}^3$$

Now pressure, $p = \rho_0 \times g \times Z$

$$= 900 \times 9.81 \times 0.4 = 3531.6\ \frac{\text{N}}{\text{m}^2} = \frac{3531.6}{10^4}\frac{\text{N}}{\text{cm}^2}.$$

$$= \mathbf{0.35316\,\frac{N}{cm^2}}.$$

(c) For mercury, specific gravity $= 13.6$

Again, We know that the density of a fluid is the product of its specific gravity and the density of water.

∴ Density of mercury,

$$\rho_s = \text{ Specific gravity of mercury } \times \text{ Density of water}$$

$$= 13.6 \times 1000 = 13600\ \text{kg/m}^3$$

Therefore, $p = \rho_s \times g \times Z$

$$= 13600 \times 9.81 \times 0.4 = 53366.4\ \frac{\text{N}}{\text{m}^2}$$

$$= \frac{53366.4}{10^4} = \mathbf{5.34\,\frac{N}{cm^2}}.$$

Problem 4.5.4 The pressure intensity at a point in a fluid is $4.924\ \text{N/cm}^2$. Determine the corresponding height of the fluid when the fluid is:

(a) water, and (b) oil of specific gravity 0.7.

Solution. Pressure intensity,

$$p = 4.924\frac{\text{N}}{\text{cm}^2} = 4.924 \times 10^4\ \frac{\text{N}}{\text{m}^2}.$$

The corresponding height, Z, of the fluid is determined using equation (4.5) as follows:

$$\text{Z} = \frac{p}{\rho \times g}$$

(a) For water, $\rho = 1000\ \text{kg/m}^3$

Therefore, $\text{Z} = \frac{p}{\rho \times g} = \frac{4.924\times10^4}{1000\times9.81} = 5.02\ \text{m}$ of water.

(b) For oil, specific gravity $= 0.7$

Density of oil $\rho_0 = 0.7 \times 1000 = 700\ \text{kg/m}^3$

Therefore, $Z = \frac{p}{\rho_0 \times g} = \frac{4.924\times10^4}{700\times9.81} = 7.17\ \text{m}$ of oil.

Problem 4.5.5 An oil with a specific gravity of 0.7 is contained in a vessel. At a certain point, the height of the oil is 50 m. Determine the corresponding height of water at that point.

Solution. Specific gravity of oil, $S_0 = 0.7$

Height of oil, $Z_0 = 50\ m$

Density of oil, $\rho_0 = \text{Specific gravity of oil} \times \text{Density of water}$

$$= 0.7 \times 1000 = 700\ \text{kg/m}^3$$

Intensity of pressure, $p = \rho_0 \times g \times Z_0 = 700 \times 9.81 \times 50 \frac{\text{N}}{\text{m}^2}$

$$\text{Corresponding height of water} = \frac{p}{\text{Density of water } \times g}$$

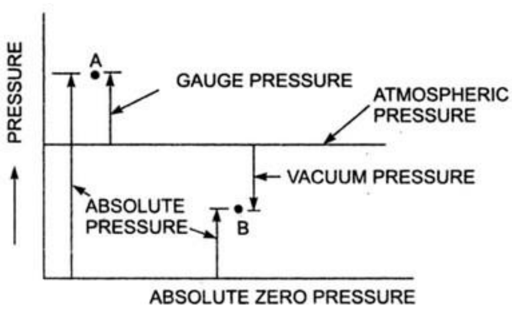
$$= \frac{700 \times 9.81 \times 50}{1000 \times 9.81} = 0.7 \times 50 = \mathbf{35\ m} \text{ of water.}$$

4.6 ABSOLUTE, GAUGE AND VACUUM PRESSURES

Fluid pressure is typically measured using two systems: one measures pressure relative to absolute zero or complete vacuum, known as absolute pressure, while the other measures pressure relative to atmospheric pressure, known as gauge pressure. Therefore:

1. **Absolute pressure** is the pressure measured relative to absolute vacuum pressure.
2. **Gauge pressure** is the pressure measured using a pressure measuring instrument where atmospheric pressure is taken as the reference point, marked as zero on the scale.
3. **Vacuum pressure** is the pressure measured below atmospheric pressure.

Figure 4.4 illustrates the relationship between absolute pressure, gauge pressure, and vacuum pressure.



Mathematically we can write,

(i) Absolute pressure = Atmospheric pressure + Gauge pressure

or $p_{\text{ab}} = p_{\text{atm}} + p_{\text{gauge}}$ ***Figure 4.4: Relationship between pressures.***

(ii) Vacuum pressure = Atmospheric pressure − Absolute pressure.

Problem 4.6.1 What are the gauge pressure and absolute pressure at a point 5 m below the free surface of a liquid with a density of $1.63 \times 10^3\ \text{kg/m}^3$, given that the atmospheric pressure equals 750 mm of mercury? The specific gravity of mercury is 13.6 and the density of water is $1000\ \text{kg/m}^3$.

Solution. Depth of liquid, $Z_1 = 5\ m$.

Density of liquid, $\rho_1 = 1.63 \times 10^3\ \text{kg/m}^3$.

Atmospheric pressure head, $Z_0 = 750$ mm of Hg.

$$= \frac{750}{1000} = 0.75 \text{ m of Hg}$$

Therefore, atmospheric pressure, $p_{\text{atm}} = \rho_0 \times g \times Z_0$,

where ρ_0 = Density of Hg = Special gravity of mercury × Density of water = $13.6 \times 1000\ \text{kg/m}^3$

and Z_0 = Pressure head in terms of mercury.

$$\therefore\ p_{\text{atm}} = (13.6 \times 1000) \times 9.81 \times 0.75\ \text{N/m}^2\ (\because Z_0 = 0.75)$$

$$= 100062\ \text{N/m}^2$$

The pressure at a point located 5 m below the free surface of the liquid is expressed as

$$p = \rho_1 \times g \times Z_1$$

$$= (1.63 \times 1000) \times 9.81 \times 5 = 79951.5\ \text{N/m}^2.$$

Therefore, Gauge pressure, $p = \mathbf{79951.5\ N/m^2}$.

Now absolute pressure = Gauge pressure + Atmospheric pressure

$$= 79951.5 + 100062 = \mathbf{180013.5\ N/m^2}.$$

4.7 FLUIDS UNDER RIGID BODY MOTION

In certain cases of fluid flow, the characteristics of fluids in motion can be understood through principles of hydrostatics. Fluids in such motion are considered to be in a state of relative equilibrium or relative rest. This occurs when a fluid flows with a constant velocity without acceleration, or with uniform acceleration.

4.7.1 STATIC FLUID SUBJECT TO UNIFORM ACCELERATION

When a fluid moves uniformly in a straight line without acceleration, there are no shear forces or inertia forces acting on the fluid particles, which continue their motion due to inertia alone. Each fluid particle's weight is balanced by pressure forces, akin to a stationary fluid mass, allowing the application of hydrostatic equations without modification. If the entire fluid undergoes uniform acceleration in a straight line without relative movement between layers, there are still no shear forces, but an additional force acts to cause acceleration. However, with appropriate adjustment for this additional force, the system can be analyzed using hydrostatic methods. Consider a rectangular fluid element in a three-dimensional Cartesian coordinate system, as illustrated in Figure 4.5. The pressure at the center of the element is p. The fluid element moves with constant acceleration, with components a_x, a_y and a_z along the coordinate axes *x*, *y*, and *z*, respectively. The force acting on the fluid element in the *x*-direction is

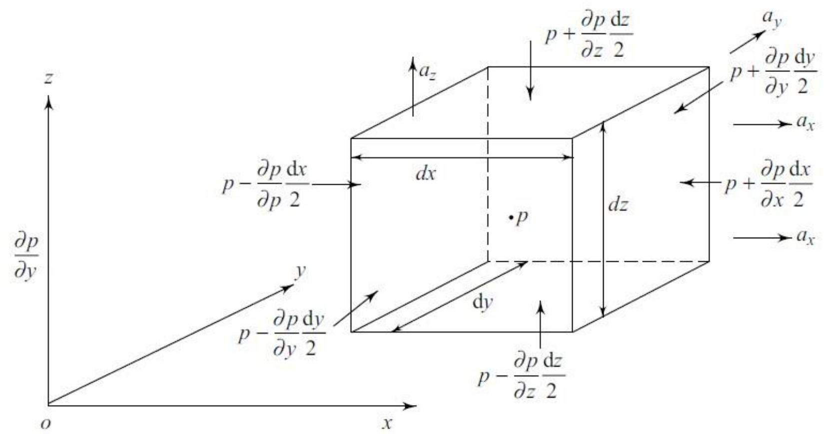


Figure 4.5: Equilibrium of fluid element moving with constant acceleration.

$$\left[\left(p - \frac{\partial p}{\partial x}\frac{1}{2}\,\mathrm{d}x\right) - \left(p + \frac{\partial p}{\partial x}\frac{1}{2}\,\mathrm{d}x\right)\right]\mathrm{d}y\;\mathrm{d}z$$
$$= \frac{\partial p}{\partial x} dxdy\;\mathrm{d}z$$

Therefore the equation of motion in the x direction can be written as

$$\rho \mathrm{d}x\;\mathrm{d}y\;\mathrm{d}z a_x = -\frac{\partial p}{\partial x}\;\mathrm{d}x\;\mathrm{d}y\;\mathrm{d}z$$

or

$$\frac{\partial p}{\partial x} = -\rho a_x \tag{4.6}$$

where ρ is the density of the fluid. In a similar fashion, the equation of motion in the y direction can be written as

$$\frac{\partial p}{\partial y} = -\rho a_y \tag{4.7}$$

The total force on the fluid element in the z direction is the result of the difference between the pressure force and the weight. Thus, the equation of motion in the z direction can be expressed as

$$\left(-\frac{\partial p}{\partial z} - \rho g\right)\mathrm{d}x\;\mathrm{d}y\;\mathrm{d}z = \rho a_z\;\mathrm{d}x\;\mathrm{d}y\;\mathrm{d}z$$

or

$$\frac{\partial p}{\partial z} = -\rho(g + a_z) \tag{4.8}$$

It is noted that the governing equations for pressure distribution (Eqs (4.6), (4.7), and (4.8)) resemble those seen in hydrostatic pressure equations. In a simplified two-dimensional scenario where the y

component of acceleration, a_y, is zero, a surface of constant pressure in the fluid will be defined by

$$\mathrm{d}p = \frac{\partial p}{\partial x}\,\mathrm{d}x + \frac{\partial p}{\partial z}\,\mathrm{d}z = 0$$

$$\frac{\mathrm{d}z}{\mathrm{d}x} = -\frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial z}} = -\frac{a_x}{g + a_z}$$

Given that a_x and a_z are constants, a surface of constant pressure exhibits a uniform slope. One such surface is a free surface, where $p = p_{\text{atm}}$. Other planes of constant pressure are parallel to it.

4.7.1.1. ACCELERATION IN HORIZONTAL DIRECTION

For $a_z = 0$, we have

$$\frac{dz}{dx} = -\frac{a_x}{g}$$

The negative sign indicates that the surface slopes downward when $a_x > 0$, indicating that the container is accelerating. Conversely, if $a_x < 0$, indicating deceleration, the surfaces will have a positive slope. These

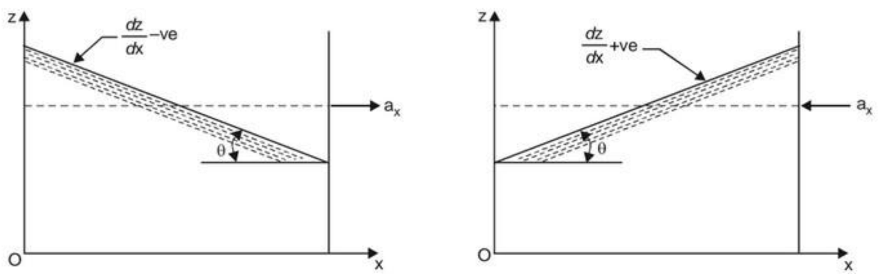


Figure 4.6: Free surface during horizontal acceleration.

two scenarios are illustrated in Figure 4.6.

If θ is the angle made by the free surface with horizontal. Then

$$\tan\theta = \frac{a_x}{g}$$

4.7.1.2. ACCELERATION IN VERTICAL DIRECTION

For $a_x = 0$ we have $\frac{dz}{dx} = 0$, and the surfaces of constant pressure are horizontal. The pressure gradient,

$$\frac{dp}{dz} = -\gamma\left(1 + \frac{a_z}{g}\right),$$

where the negative sign indicates pressure decreases with increasing z. If distance h is measured vertically downwards from the free surface, then

$$\frac{dp}{dh} = \gamma\left(1 + \frac{a_z}{g}\right)$$

Upon integration, we have

$$p = \gamma\left(1 + \frac{a_z}{g}\right)h + C$$

At the free surface, $h = 0, p = 0$. Therefore, $C = 0$. Thus

$$p = \gamma\left(1 + \frac{a_z}{g}\right)h$$

If $a_z > 0$, then $(p - p_{\text{hyd}}) = \frac{\gamma a_z h}{g}$

If $a_z < 0$, then $(\text{p} - p_{\text{hyd}}) = -\frac{\gamma a_z h}{g}$

If $a_z = -g$ (representing freely falling fluid), then $p = 0$ (indicating atmospheric pressure). These observations are depicted graphically in Figure 4.7.

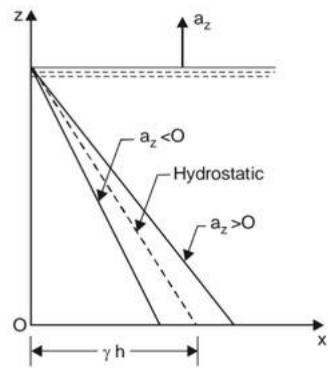


Figure 4.7: Pressure variation in the fluid subject to verticle accelaration.

Example 4.7.1 A closed container filled with water to a depth of 3 m is moving vertically downward with an acceleration of 4 $\mathrm{m/s^2}$. Determine the pressure at the bottom of the container.

Solution.

Given: $h = 3$ m, $a_z = -4\ \mathrm{m/s^2}$

$$p = \gamma\left(1 + \frac{a_z}{g}\right)h$$

$$= 9810\left(1 - \frac{4}{9.81}\right) \times 3$$

$$= 17430\ \mathrm{N/m^2}$$

Example 4.7.2 An oil tanker with a cross-section of 2.4 m × 2.4 m and a length of 4 m is filled with oil to a depth of 1.5 m. It moves with uniform acceleration, causing the front bottom corner to be exposed. Calculate the total horizontal force acting on the sides of the tanker. Assume the specific gravity of the oil is 0.85.

Solution. Volume of oil inside the tanker, $V = 2.4 \times 1.5 \times 4 = 14.4\ \mathrm{m^3}$

Volume of empty space inside the tanker

$$= 0.9 \times 2.4 \times 4 = 8.64\ \mathrm{m^3}$$

$$\tan\theta = \frac{AB}{BF} = \frac{2.4}{x}$$

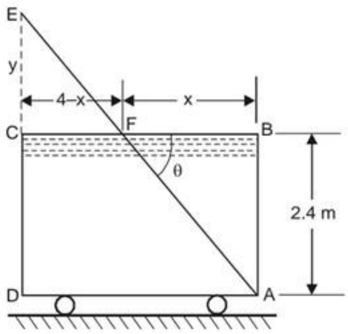


Figure 4.8

Volume of empty space

$$= \frac{1}{2} \times AB \times BF \times 2.4$$

$$= \frac{1}{2} \times 2.4 \times x \times 2.4 = 2.88x$$

$$2.88x = 8.64$$

$$x = 3\ \mathrm{m}$$

$$\tan\theta = \frac{a_x}{g} = \frac{2.4}{x}$$

$$a_x = \frac{2.4}{3} \times 9.81 = 7.848\ \mathrm{m/s^2}$$

$\triangle\, AED$ and $\triangle\, FEC$ are similar. Therefore,

$$\frac{AD}{DE} = \frac{CF}{CE}$$
$$\frac{4}{2.4 + y} = \frac{4 - 3}{y}$$
$$4y = 1(2.4 + y)$$
$$3y = 2.4$$
$$y = 0.8 \text{ m}$$

Hydrostatic force on side $CD = \gamma h_c A = 0.85 \times 9810 \times \left(0.8 + \frac{2.4}{1.5}\right) \times 2.4 \times 2.4 = 115271.42$ N.

Example 4.7.3 In an open rectangular tank measuring 5 m in length and 2.5 m in width, water is filled to a depth of 1.5 m. Determine the slope of the water surface when the tank accelerates upward at 3 m/s^2 along an inclined plane of 45°.

Solution.

Given that $\alpha = 30°, \text{a} = 3 \text{ m/s}^2$

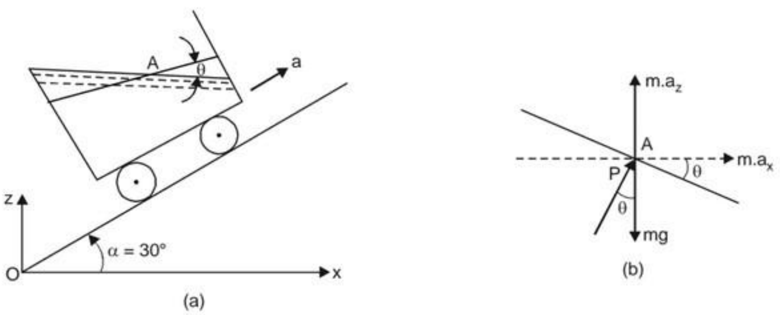


Figure 4.9

Let $P =$ Pressure force acting normal to the water surface.

Then

$$P\sin\theta = m \cdot a_x$$
$$P\cos\theta = m \cdot a_z + mg$$
$$\tan\theta = \frac{a_x}{a_z + g} = \frac{a\cos\alpha}{a\sin\alpha + g}$$
$$= \frac{3\cos 45°}{3\sin 45° + 9.81} = 0.1778$$
$$\theta = 10.08°.$$

4.8 SUMMARY

In this unit we studied:

(i) The definition of fluid pressure at a point.

(ii) Pascal's law.

(iii) Pressure variation in a fluid at rest.

(iv) The difference between absolute, gauge and vaccum pressure.

(v) Pressure in a fluid under uniform acceleration in horizontal and vertical direction.

4.9 GLOSSARY

(i) Fluid pressure

(ii) Pascal's law

(iii) Hydrostatic Law

(iv) Pressure head

(v) Absolute pressure

(vi) Gauge pressure

(vii) Vaccum pressure

4.10 REFERENCES AND SUGGESTED READINGS

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4.11 TERMINAL QUESTIONS

1. State and prove Pascal's law.
2. Explain hydrostatic law.
3. Differentiate between absolute pressure, gauge pressure and vaccum pressure.
4. An open tank contains water upto a depth of 2m and above it an oil of specific gravity 0.9 for a depth of 1m. Find the pressure intensity (i) at the interface of the two liquids (ii) at the bottom of the tank.

   [**Ans.**(i)0.8829N/cm^2 (ii)2.8449 N/cm^2]
5. A hydraulic press has a plunger of diameter 4 cm and a ram of diameter 20cm. It is used for lifting a weight of 20kN. Find the force required at the plunger. [**Ans.** 800N]
6. Determine the absolute and gauge pressure at a point that is 2 m below the free surface of water. Take atmospheric pressure as 10.1043 N/cm^2.

   [**Ans.** 1.962 N/cm^2 (gauge), 12.066 N/cm^2(abs.)]

UNIT 5: IMMISCIBLE FLUIDS

CONTENTS:

5.1 INTRODUCTION

Immiscible liquids are liquids that don't mix with each other to form a single phase or homogeneous mixture. Instead, they form two distinct layers.

5.2 OBJECTIVES

Upon reading this unit learner will be able to:

1. Immiscible liquids.
2. Flow of two immiscible viscous fluids between two parallel plates.

5.3 IMMISCIBLE FLUIDS

The fluids which don't mix or are not soluble in each other, are called immiscible fluids. For example, Petrol and water are two immiscible fluids. The top layer will be formed by petrol since it is lighter than water. Any two immiscible fluids when mixed form an emulsion.

Two immiscible fluids can be separated by separating funnel. We put two immiscible fluids into the funnel and are left for a short time to settle out and form two layers. The tap of the funnel is opened and the bottom liquid is allowed to run. The two fluids are now separate.

5.4 FLOW OF TWO IMMISCIBLE VISCOUS FLUIDS BETWEEN TWO PARALLEL PLATES

Consider the flow of two immiscible fluids between two parallel fixed horizontal plates under a constant pressure gradient $P\left(=-dp/dx\right)$.

Let the fluid with a coefficient of viscosity μ_1 extend from $y=-h$ to $y=h_0$ and fluid with a coefficient of viscosity μ_2 extend from $y=h_0$ to $y=h$.

Assuming the fluids to be of constant densities and the flow to be steady uni-directional and depending on y alone, the Navier-Stokes equation in $x-direction$ is given by

$$0=-(dp/dx)+\mu\left(du^2/dy^2\right) \; or \; d^2u/dy^2=-(P/\mu) \tag{5.1}$$

Integrating Eq. (5.1), $$du/dy=-(P/\mu)y+c_1; \; c_1=const. \tag{5.2}$$

Integrating Eq. (5.2), $$u=-(P/2\mu)y^2+c_1y+c_2; \; c_2=const. \tag{5.3}$$

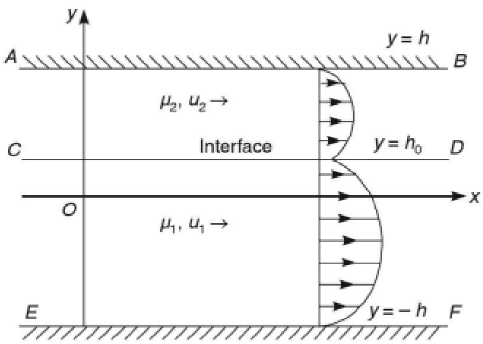

Fig. 5.1

Let u_1 be the velocity in the region $-h<y<h_0$ and u_2 be the velocity in the region $h_0<y<h$. Then, from Eq. (5.3), we get

$$u_1=-(P/2\mu_1)y^2+Ay+B, \;\; where \; -h<y<h_0 \tag{5.4}$$

$$u_2=-(P/2\mu_2)y^2+Cy+D, \;\; where \; h_0<y<h \tag{5.5}$$

where, the arbitrary constants $A, B, C \; and \; D$ are to be obtained from the boundary and interface conditions. The boundary conditions for $u_1 \, and \, u_2$ are given by

$$u_1=0, \;\; where \; y=-h \tag{5.6}$$

and $$u_2=0, \;\; where \; y=h \tag{5.7}$$

Since both the fluids are taken to be viscous, the fluids cannot slip over each other and consequently the velocity must be continuous at the interface. Thus,

$$u_1 = u_2, \quad at \;\; y = h_0 \tag{5.8}$$

By balancing the forces on a fluid element partly in the first fluid and partly in the other, it follows that the shearing stress must be continuous at the interface. Thus, we have

$$\mu_1 \left(du_1/dy \right) = \mu_2 \left(du_2/dy \right), \quad at \;\; y = h_0 \tag{5.9}$$

In view of Eq. (5.6), putting $u_1 = 0 \, and \; y = -h$ in Eq. (5.4), we have

$$0 = -\left(P/2\mu_1 \right) \times h^2 - Ah + B \tag{5.10}$$

Subtracting Eq. (5.10) from Eq. (5.4), we have

$$u_1 = -\left(P/2\mu_1 \right) \times \left(y^2 - h^2 \right) + A\left(y + h \right), where \; -h < y < h_0 \tag{5.11}$$

In view of Eq. (5.7), putting $u_2 = 0 \, and \; y = h$ in Eq. (5.5), we have

$$0 = -\left(P/2\mu_2 \right) \times h^2 + Ch + D \tag{5.12}$$

Subtracting Eq. (5.12) from Eq. (5.5), we have

$$u_2 = -\left(P/2\mu_2 \right) \times \left(y^2 - h^2 \right) + C\left(y - h \right), where \; h_0 < y < h \tag{5.13}$$

Putting $y = d_0$ in Eqs. (5.11) and (5.13) and equating the values so obtained by virtue of Eq. (5.8), we get

$$-\left(P/2\mu_1 \right) \times \left(h_0^{\,2} - h^2 \right) + A\left(h_0 + h \right) = -\left(P/2\mu_2 \right) \times \left(h_0^{\,2} - h^2 \right) + C\left(h_0 - h \right)$$

$$\Rightarrow \; C\left(h_0 - h \right) - A\left(h_0 + h \right) = \left\{ P\left(h_0^{\,2} - h^2 \right)/2 \right\} \times \left(1/\mu_1 - 1/\mu_2 \right) \tag{5.14}$$

From Eqs. (5.11) and (5.13), we have

$$du_1/dy = -\left(P/2\mu_1 \right) \times 2y + A, where \; -h < y < h_0 \tag{5.15}$$

and $$du_2/dy = -\left(P/2\mu_2 \right) \times 2y + C, where \; h_0 < y < h \tag{5.16}$$

Putting $y = h_0$ in Eqs. (5.15) and (5.16) and substituting the values so obtained in Eq. (5.9), we obtain

$$\mu_1 \{-(P/2\mu_1)\times 2h_0 + A\} = \mu_2 \{-(P/2\mu_2)\times 2h_0 + C\} \text{ so that } \mu_1 A = \mu_2 C \quad (5.17)$$

Eliminating C from Eqs. (5.14) and (5.17), we get

$$(\mu_1/\mu_2)\times A(h_0 - h) - A(h_0 + h) = \{P(h^2 - h_0^{\,2})/2\}\times(1/\mu_1 - 1/\mu_2)$$

$$\Rightarrow \frac{A\mu_1(h_0 - h) - A\mu_2(h_0 + h)}{\mu_2} = \frac{P(h^2 - h_0^{\,2})}{2}\times\frac{\mu_2 - \mu_1}{\mu_1\mu_2}$$

$$\Rightarrow A\{-h_0(\mu_2 - \mu_1) - h(\mu_2 + \mu_1)\} = \frac{P(h^2 - h_0^{\,2})}{2}\times\frac{\mu_2 - \mu_1}{\mu_1}$$

$$\Rightarrow -A\left\{h + h_0\left(\frac{\mu_2 - \mu_1}{\mu_2 + \mu_1}\right)\right\} = \frac{P(h^2 - h_0^{\,2})}{2}\times\frac{\mu_2 - \mu_1}{\mu_2 \times \mu_1}$$

$$\Rightarrow A = -\{(P/2\mu_1)\times(h^2 - h_0^{\,2})\alpha\}/(h + h_0\alpha), \quad (5.18)$$

where $\alpha = (\mu_2 - \mu_1)/(\mu_2 - \mu_1)$ (5.19)

Substituting the value of A given by Eq. (5.18) in Eq. (5.11), we have

$$u_1 = -\frac{P}{2\mu_1}\left\{y^2 - h^2 + \frac{\alpha(h^2 - h_0^{\,2})(y + h)}{h + \alpha h_0}\right\}, \text{ where } -h < y < h_0 \quad (5.20)$$

Substituting the value of A given by Eq. (5.18) in Eq. (5.17),

$$C = -\frac{P}{2\mu_2}\times\frac{\alpha(h^2 - h_0^{\,2})}{h + \alpha h_0} \quad (5.21)$$

Substituting the above value of C in Eq. (5.13), we have

$$u_2 = -\frac{P}{2\mu_2}\left\{y^2 - h^2 + \frac{\alpha(h^2 - h_0^{\,2})(y - h)}{h + \alpha h_0}\right\}, \text{ where } h_0 < y < h \quad (5.22)$$

The required velocity is given by Eq. (5.20) and Eq. (5.22), The velocity distribution is plotted in *Fig. 5.1*. Observe that at the interface CD the slope of the velocity profile is discontinuous. This is so due to change in the coefficients of viscosity of the fluids on the two side of the interface.

The flux Q is given by $Q = \int_{-h}^{h_0} u_1 dy + \int_{h_0}^{h} u_2 dy$ (5.23)

Substituting the values of u_1 *and* u_2 given by Eq. (5.20) and Eq. (5.22), respectively in Eq. (5.23) and simplifying, we have

$$Q = \frac{P(\mu_1 + \mu_2)}{6\mu_1\mu_2}\left\{2h^3 - \alpha\left(h_0^{\,3} - 2h^2h_0\right) - \beta\left(2hh_0 + \alpha h^2 + \alpha h_0^{\,2}\right)\right\}, \qquad (5.24)$$

where $\beta = 3\alpha\left(h^2 - h_0^{\,2}\right)/2\left(h + \alpha h_0\right)$ (5.25)

Particular case (i) If $\mu_1 = \mu_2 = \mu(say)$, then Eqs. (5.19) and (5.25), yield $\alpha = \beta = 0$. hence, Eq. (5.24) reduces to

$$Q = \left(2Ph^3\right)/3\mu,$$

which is the flux obtained in the case of Poiseuille flow.

Particular case (ii) If we assume that the whole space is filled with fluid of coefficient of viscosity μ_1 so that $h_1 = h$. Then, we get $Q = \left(2Ph^3\right)/3\mu_1$.

Similarly, if we take $h_0 = -h$, we get $Q = \left(2Ph^3\right)/3\mu_2$.

Example 1: Two fluids of coefficient of viscosities μ_1 and μ_2 confined in region $-d < y < 0$ and $0 < y < d$ respectively, are flowing between two parallel plates under a constant pressure gradient $P\left(= -\partial p/\partial x\right)$. Show that when the plate at $y = d$ is moving with constant velocity U, then the velocity distribution is given by

$$u = \frac{P}{2\mu_1}\left(d^2 - y^2\right) + \frac{\varepsilon\mu_2 U}{(\mu_1 + \mu_2)d}\left(y + d\right), -d < y < 0$$

$$u = U + \frac{P}{2\mu_2}\left(d^2 - y^2\right) + \frac{\varepsilon\mu_1 U}{(\mu_1 + \mu_2)d}\left(y - d\right), 0 < y < d, where\, \varepsilon = 1 + \frac{1}{2}\frac{Pd^2(\mu_1 - \mu_2)}{\mu_1\mu_2 U}.$$

Solution. Here, we consider the flow of two immiscible fluids between two parallel plates $AB\,(y=-d)$ at rest and $CD\,(y=d)$ moving with constant velocity U under a constant pressure gradient $P(=-\partial p/\partial x)$.

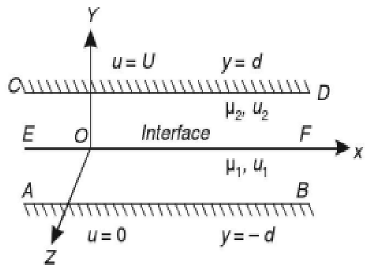

Fig. 5.2

Let be the direction of flow, y the direction perpendicular to the flow, and the width of the plates parallel to the $z-direction$. Here, the word infinite implies that the width of the plates is large compared with $2d$ and the flow may be treated as two-dimensional (i.e., $\partial/\partial z=0$). Let the plates be long enough in the $x-direction$ for the flow to be parallel. Here, we take the velocity components $v\,and\,w$ to be zero everywhere. Moreover, the flow being steady, variables are independent of time $(\partial/\partial t=0)$. Furthermore, the equation of continuity (namely $\partial u/\partial x+\partial v/\partial y+\partial w/\partial z=0$) reduces to $\partial u/\partial x=0$ so that $u=u(y)$. Thus, for the present problem, we have

$$u=u(y),\ v=0,\ w=0,\ \partial/\partial z=0,\ \partial/\partial t=0 \tag{1}$$

For the present two-dimensional flow in the absence of body forces, the Navier-stokes equations for $x\,and\,y$ directions are given by

$$0=-\partial p/\partial x+\mu\left(\partial^2 u/\partial y^2\right) \tag{2}$$

$$\text{and } 0=-\partial p/\partial y \tag{3}$$

Equation (3) shows that the pressure does not depend on y. Hence, p is a function of x alone and so Eq. (2) reduces to $d^2u/dy^2=(1/\mu)\times(dp/dx)$.

Since $P=-dp/dx$, the above equation reduces to $d^2u/dy^2=(-P/\mu)$. (4)

Integrating Eq. (4), $du/dy=-(P/\mu)\,y+C_1;\ C_1=const.$ (5)

Integrating Eq. (5), $u=-(P/2\mu)\,y^2+C_1y+C_2;\ C_2=const.$ (6)

Let u_1 be the velocity in the region $-d<y<0$ and u_2 be the velocity in the region $0<y<d$, then Eq. (6) gives

$$u_1 = -(P/2\mu_1)y^2 + Ay + B;\ when -d < y < 0 \tag{7}$$

and $$u_2 = -(P/2\mu_2)y^2 + Cy + D;\ when\, 0 < y < d \tag{8}$$

Here $A, B, C\ and\ D$ are arbitrary constants. We now proceed to calculate these constants with help of boundary conditions and interface conditions.

Now, boundary conditions for $u_1\ and\ u_2$ are given by

$$u_1 = 0,\quad where\ y = -d \tag{9}$$

and $$u_2 = U,\quad where\ y = d \tag{10}$$

Putting $u_1 = 0$ and $y = -d$ in Eq. (7), we have

$$0 = -(P/2\mu_1)d^2 - Ad + B, \tag{11}$$

Subtracting Eq. (11) from Eq. (7), we obtain

$$u_1 = -(P/2\mu_1)(y^2 - d^2) + A(y + d);\ when -d < y < 0 \tag{12}$$

Next, putting $u_2 = U$ and $y = d$ in Eq. (8), we have

$$U = -(P/2\mu_2)d^2 + Cd + D. \tag{13}$$

Subtracting Eq. (13) from Eq. (8), we obtain

$$u_2 - U = -(P/2\mu_2)(y^2 - d^2) + C(y - d);\ when\, 0 < y < d. \tag{14}$$

Since both the fluids are taken to be viscous, the fluids cannot slip over each other and hence the velocity has to be continuous at the interface EF. Thus,

$$u_1 = u_2,\quad at\ y = 0. \tag{15}$$

Putting in Eqs. (12) and (14) and equating the values so obtained by virtue of Eq. (15) gives

$$(P/2\mu_1)\times d^2 + Ad = (P/2\mu_2)\times d^2 - Cd + U \text{ or } d(A + C) = (Pd^2/2)\times(1/\mu_2 - 1/\mu_1) + U$$

$$\Rightarrow \; A+C=\left(\frac{Pd}{2}\right)\times\left(\frac{1}{\mu_2}-\frac{1}{\mu_1}\right)+\frac{U}{d} \tag{16}$$

Further, by balancing the forces on a fluid element partly lying in the first fluid and partly in the other, we find that the shearing stress has to be continuous at the interface EF. Thus,

$$\mu_1\left(du_1/dy\right)=\mu_2\left(du_2/dy\right), \quad at \; y=0 \tag{17}$$

From Eq. (12), $du_1/dy=-\left(Py/\mu_1\right)+A$

When $y=0$, $du_1/dy=A$ (18)

Now, from Eq. (142), $du_2/dy=-\left(Py/\mu_2\right)+C$

When $y=0$, $du_2/dy=C$ (19)

Using Eqs. (18) and (19), Eq. (17) implies that

$$\mu_1 A=\mu_2 C \;\Rightarrow\; A/\mu_2=C/\mu_1$$

$$\therefore \; \frac{A}{\mu_2}=\frac{C}{\mu_1}=\frac{A+C}{\mu_1+\mu_2}=\frac{1}{\mu_1+\mu_2}\left\{\frac{U}{d}+\frac{Pd}{2}\left(\frac{1}{\mu_2}-\frac{1}{\mu_1}\right)\right\}, \text{ using Eq. (16)}$$

$$\Rightarrow \; \frac{A}{\mu_2}=\frac{C}{\mu_1}=\left\{\frac{U}{d\left(\mu_1+\mu_2\right)}+\frac{Pd}{2\mu_1\mu_2}\frac{\left(\mu_1-\mu_2\right)}{\left(\mu_1+\mu_2\right)}\right\}$$

$$\Rightarrow \; \frac{A}{\mu_2}=\frac{C}{\mu_1}=\left\{\frac{U}{d\left(\mu_1+\mu_2\right)}+\frac{Pd}{2\mu_1\mu_2}\frac{\left(\mu_1-\mu_2\right)}{\left(\mu_1+\mu_2\right)}\right\}$$

$$\Rightarrow \; A=\frac{U\mu_2}{d\left(\mu_1+\mu_2\right)}+\frac{Pd}{2\mu_1}\frac{\left(\mu_1-\mu_2\right)}{\left(\mu_1+\mu_2\right)}, \; and \;\; C=\frac{U\mu_1}{d\left(\mu_1+\mu_2\right)}+\frac{Pd}{2\mu_2}\frac{\left(\mu_1-\mu_2\right)}{\left(\mu_1+\mu_2\right)}$$

Substituting these values of A and C in Eqs. (12) and (14), the required velocity distribution is

$$u_1=-\frac{P}{2\mu_1}\left(y^2-d^2\right)+\left(y+d\right)\left\{\frac{U\mu_2}{d\left(\mu_1+\mu_2\right)}+\frac{Pd\left(\mu_1-\mu_2\right)}{2\mu_1\left(\mu_1+\mu_2\right)}\right\}, \; when \; -d<y<0$$

and $$u_2=U-\frac{P}{2\mu_2}\left(y^2-d^2\right)+\left(y-d\right)\left\{\frac{U\mu_1}{d\left(\mu_1+\mu_2\right)}+\frac{Pd\left(\mu_1-\mu_2\right)}{2\mu_2\left(\mu_1+\mu_2\right)}\right\}, \; when \; 0<y<d$$

Rewriting the above results, we have

$$u_1 = \frac{P}{2\mu_1}\left(d^2 - y^2\right) + \left[1 + \frac{Pd^2(\mu_1 - \mu_2)}{2\mu_1\mu_2 U}\right]\frac{U\mu_2(y+d)}{d(\mu_1+\mu_2)} \tag{20}$$

and $$u_2 = U + \frac{P}{2\mu_2}\left(d^2 - y^2\right) + \left[1 + \frac{Pd^2(\mu_1 - \mu_2)}{2\mu_1\mu_2 U}\right]\frac{U\mu_1(y-d)}{d(\mu_1+\mu_2)} \tag{21}$$

Given that $\varepsilon = 1 + \left\{Pd^2(\mu_1 - \mu_2)\right\}/(2\mu_1\mu_2 U)$. Hence, the required velocity distribution given by Eqs. (20) and (21) can be re-written as

$$u = \frac{P}{2\mu_1}\left(d^2 - y^2\right) + \frac{\varepsilon\mu_2 U}{d(\mu_1+\mu_2)}(y+d), \qquad -d < y < 0$$

$$= U + \frac{P}{2\mu_2}\left(d^2 - y^2\right) + \frac{\varepsilon\mu_1 U}{d(\mu_1+\mu_2)}(y-d), \quad 0 < y < d.$$

5.5 *SUMMARY*

This unit explains the following topics:

(i) Definition of Fluid.

(ii) Real and Ideal Fluids.

(iii) Definition of Newtonian and non-Newtonian fluids based on Newton's law of viscosity.

(iv) Different types of non-Newtonian fluids.

(v) Continuum hypothesis.

(vi) Velocity and acceleration of fluid particle.

5.6 *GLOSSARY*

(i) Fluid

(ii) Viscosity

(iii) Newton's law of viscosity

(iv) Newtonian and non-Newtonian fluids

(v) Shear stress

5.7 *REFERENCES AND SUGGESTED READINGS*

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(iv) P.K. Kundu, I.M. Cohen & D.R. Dowling (2015), Fluid Mechanics, Academic Press; 6th edition.

(v) F.M. White & H. Xue (2022), Fluid Mechanics, McGraw Hill; Standard Edition.

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5.8 *TERMINAL QUESTIONS*

1. Immiscible fluids oil and water can be
   (i) Separated by use of funnel.
   (ii) Can not be separated.
   (iii) Separated by use of a separating funnel.
   (iv) None of these.
2. Two liquids that can be mixed together but separate shortly after are:
   (i) immiscible
   (ii) insoluble
   (iii) miscible
   (iv) soluble

3. The constant pressure gradient P for the flow of two immiscible fluids between two fixed parallel horizontal plates is
   (i) $P = 0$

(ii) $P = -\dfrac{dp}{dx}$

(iii) $P = const.$

(iv) None of these

4. If fluids are to be of constant densities and the flow be steady uni-directional and depending on y alone, the Navier-Stokes equation in $x-direction$ is given by

   (i) $d^2u/dy^2 = -(P/\mu)$

   (ii) $-(dp/dx) + \mu(du^2/dy^2) = 0$

   (iii) Both (i) and (ii)

   (iv) None of these

5. In case of a steady flow, which of the following conditions is true?

   (i) The velocity is constant in the flow field with respect to space.

   (ii) The velocity is constant at a point with respect to time.

   (iii) The velocity does not change from place to place.

   (iv) The velocity changes at a point with respect to time.

6. For a steady flow, the values of all fluid properties at any fixed point

   (i) change with location.

   (ii) do not change with time.

   (iii) do not change with location.

   (iv) change with time.

7. Two fluids are taken viscous and they cannot slip over each other. Then, velocity must be

   (i) continuous at the interface.

   (ii) zero at the interface.

   (iii) discontinuous at the interface.

   (iv) None of these.

8. If both viscosities μ_1 *and* μ_2 are equal i.e., $\mu_1 = \mu_2 = \mu(say)$. Then, the flux for the flow of two immiscible viscous fluids between two parallel plates is given by
   (i) $Q = 0$
   (ii) $Q = -(2Ph^3)/3\mu$
   (iii) $Q = (2Ph^3)/3\mu$
   (iv) $Q = 2P/3\mu h^3$

9. The flux for Poiseuille flow is given by
   (i) $Q = (2Ph^3)/3\mu$
   (ii) $Q = (2P)/(3\mu h^3)$
   (iii) $Q = 2P/\mu h^3$
   (iv) $Q = P/3\mu h^3$

10. If the flow of two immiscible fluids between two parallel plates $AB\ (y = -d)$ at rest and $CD\ (y = d)$ moving with constant velocity U under a constant pressure gradient. The flow is steady (variables are independent of time $(\partial/\partial t = 0)$). Then, the equation of continuity is given by
   (i) $\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$
   (ii) $\partial u/\partial x - \partial v/\partial y + \partial w/\partial z = 0$
   (iii) $\partial u/\partial x - \partial v/\partial y - \partial w/\partial z = 0$
   (iv) $\partial u/\partial x = 0$

11. What is the unit of rate of flow of discharge?
   (i) m^2/sec
   (ii) m^3/sec
   (iii) Litres-sec
   (iv) All of the above

12. The unit of viscosity is

(i) m^2/sec

(ii) kg-sec/m

(iii) N-sec/m^2

(iv) N-sec/m

13. Fluid is a substance that

(i) always expands until it fills any container.

(ii) can't be subjected to shear forces.

(iii) can't remain at rest under action of any shear force.

(iv) flows.

14. Immiscible fluids are soluble in each other.

(i) True

(ii) False

15. The flow in which the velocity at any given time changes with respect to space is known as

(i) Uniform flow

(ii) Non-uniform flow

(iii) Compressible flow

(iv) Incompressible flow

16. The separation of immiscible liquids can be observed as distinct layers based on density.

(i) True

(ii) False

17. Temperature changes generally do not affect the miscibility of immiscible liquids significantly.

(i) False

(ii) True

18. Two fluids of coefficient of viscosities υ_1 and υ_2 confined in region $-d < y < 0$ and $0 < y < d$ respectively, are flowing between two parallel plates under a constant pressure gradient $P\left(=-\partial p/\partial x\right)$. Show that when the plate at $y = d$ is moving with constant velocity U, then find the velocity distribution.

19. Write a short note on immiscible fluids.

20. Derive the formula for flow of two immiscible viscous fluids between two parallel plates.

21. Write a short note on Poiseuille flow.

22. Derive the expression of flux for the Poiseuille flow.

5.9 *ANSWERS*

1. (iv)
2. (i)
3. (ii)
4. (iii)
5. (ii)
6. (ii)
7. (i)
8. (iii)
9. (i)
10. (iv)
11. (ii)
12. (ii)
13. (iii)
14. (ii)
15. (ii)
16. (i)

17. (ii)

18. $$u = \begin{cases} \dfrac{P}{2\upsilon_1}\left(d^2 - y^2\right) + \dfrac{\varepsilon \upsilon_2 U}{(\upsilon_1 + \upsilon_2)d}(y + d), -d < y < 0 \\ U + \dfrac{P}{2\upsilon_2}\left(d^2 - y^2\right) + \dfrac{\varepsilon \upsilon_1 U}{(\upsilon_1 + \upsilon_2)d}(y - d), 0 < y < d \end{cases}$$

$$\textit{where}\ \varepsilon = 1 + \frac{1}{2}\frac{Pd^2(\upsilon_1 - \upsilon_2)}{\upsilon_1 \upsilon_2 U}.$$

UNIT 6: EULER'S EQUATION OF MOTION

CONTENTS:

6.1 INTRODUCTION

Euler's Equation assumes that the forces acting on the fluid are conservative. This assumption implies that the potential energy of the fluid does not change within the flow field. It allows for the conservation of mechanical energy and simplifies the mathematical formulation of the equation.

6.2 OBJECTIVES

Upon finishing this unit, learner should be able to:

(i) Euler's Equation of Motion under Conservative Body Forces.

(ii) Applications of Bernoulli's equation.

6.3 EULER'S EQUATION OF MOTION UNDER CONSERVATIVE BODY FORCES

When a velocity potential exists (so that the motion is irrotational) and the external forces are derivable from a potential function, the equations of motion can always be integrated. Let ϕ be the velocity potential and V be the force potential. Then, by definition, we get

$$u = -\frac{\partial \phi}{\partial x}, \quad v = -\frac{\partial \phi}{\partial y}, \quad w = -\frac{\partial \phi}{\partial z}, \qquad (6.1)$$

$$X = -\frac{\partial V}{\partial x}, \quad Y = -\frac{\partial V}{\partial y}, \quad Z = -\frac{\partial V}{\partial z}, \qquad (6.2)$$

and $$\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x},\quad \frac{\partial v}{\partial z}=\frac{\partial w}{\partial y},\quad \frac{\partial w}{\partial x}=\frac{\partial u}{\partial z}. \tag{6.3}$$

Then, well known Euler's dynamical equations are

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=X-\frac{1}{\rho}\frac{\partial p}{\partial x}$$
$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=Y-\frac{1}{\rho}\frac{\partial p}{\partial y}$$
$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=Z-\frac{1}{\rho}\frac{\partial p}{\partial z}$$

Using Eqs. (6.1), (6.2) and (6.3), these can be re-written as

$$\left.\begin{aligned}
-\frac{\partial^2\phi}{\partial t\partial x}+u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}+w\frac{\partial w}{\partial x}&=-\frac{\partial V}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x}\\
-\frac{\partial^2\phi}{\partial t\partial y}+u\frac{\partial u}{\partial y}+v\frac{\partial v}{\partial y}+w\frac{\partial w}{\partial y}&=-\frac{\partial V}{\partial y}-\frac{1}{\rho}\frac{\partial p}{\partial y}\\
-\frac{\partial^2\phi}{\partial t\partial z}+u\frac{\partial u}{\partial z}+v\frac{\partial v}{\partial z}+w\frac{\partial w}{\partial z}&=-\frac{\partial V}{\partial z}-\frac{1}{\rho}\frac{\partial p}{\partial z}
\end{aligned}\right\} \tag{6.4}$$

Re-writing Eq. (6.4), we get

$$-\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial t}\right)+\frac{1}{2}\frac{\partial}{\partial x}\left(u^2+v^2+w^2\right)=-\frac{\partial V}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{6.5}$$

$$-\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial t}\right)+\frac{1}{2}\frac{\partial}{\partial y}\left(u^2+v^2+w^2\right)=-\frac{\partial V}{\partial y}-\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{6.6}$$

$$-\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial t}\right)+\frac{1}{2}\frac{\partial}{\partial z}\left(u^2+v^2+w^2\right)=-\frac{\partial V}{\partial z}-\frac{1}{\rho}\frac{\partial p}{\partial z} \tag{6.7}$$

Now $$d\left(\frac{\partial\phi}{\partial t}\right)=\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial t}\right)dx+\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial t}\right)dy+\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial t}\right)dz \tag{6.8}$$

$$dV = \left(\frac{\partial V}{\partial x}\right)dx + \left(\frac{\partial V}{\partial y}\right)dy + \left(\frac{\partial V}{\partial z}\right)dz \qquad (6.9)$$

$$dp = \left(\frac{\partial p}{\partial x}\right)dx + \left(\frac{\partial p}{\partial y}\right)dy + \left(\frac{\partial p}{\partial z}\right)dz \qquad (6.10)$$

$$d\left(u^2+v^2+w^2\right) = \frac{\partial}{\partial x}\left(u^2+v^2+w^2\right)dx + \frac{\partial}{\partial y}\left(u^2+v^2+w^2\right)dy + \frac{\partial}{\partial z}\left(u^2+v^2+w^2\right)dz \qquad (6.11)$$

Multiplying Eqs. (6.5), (6.6) and (6.7) by $dx, dy\ and\ dz$ respectively, then adding and using Eqs. (6.8), (6.9), (6.10) and (6.11), we have

$$-d\left(\frac{\partial \phi}{\partial t}\right) + \frac{1}{2}d\left(u^2+v^2+w^2\right) = -dV - \frac{1}{\rho}dp$$

$$\Rightarrow -d\left(\frac{\partial \phi}{\partial t}\right) + \frac{1}{2}dq^2 + dV + \frac{1}{\rho}dp = 0 \qquad (6.12)$$

where $q^2 = u^2+v^2+w^2 =$ (velocity of the fluid particle)^2

If ρ is a function of p, integration of Eq. (6.12) gives

$$-\frac{\partial \phi}{\partial t} + \frac{1}{2}q^2 + V + \int \frac{dp}{\rho} = F(t), \qquad (6.13)$$

where $F(t)$ is an arbitrary function of t arising from integration in which t is regarded as constant. Eq. (6.13) is Bernoulli's equation in its most general form. Equation (6.13) is also known as the pressure equation.

Case (i) Let the fluid be homogeneous and inelastic (so that $\rho =$ constant i.e., fluid is incompressible). Then, Bernoulli's equation for unsteady and irrotational motion is given by

$$-\frac{\partial \phi}{\partial t}+\frac{1}{2}q^2+V+\frac{p}{\rho}=F(t), \tag{6.14}$$

Case (ii) If the motion be steady $\frac{\partial \phi}{\partial t}=0$. Then, Bernoulli's equation for the steady irrotational motion of an incompressible fluid is given by

$$\frac{q^2}{2}+V+\frac{p}{\rho}=C, \text{ where } C \text{ is an absolute constant.} \tag{6.15}$$

6.4 CASE OF STEADY MOTION UNDER CONSERVATIVE BODY FORCES

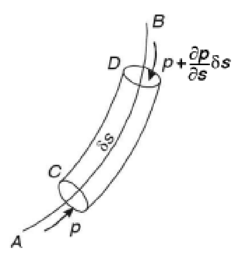
When the motion is steady and the velocity potential does not exist, we have

$$\frac{1}{2}q^2+V+\int\frac{dp}{\rho}=C,$$

where V is the force potential from which the external forces are derivable.

Proof. Consider a streamline AB in the fluid. Let δs be an element of this stream line and CD be a small cylinder of cross-sectional area α and δs as axis. If q be the velocity and S be the component of external force per unit mass in direction of the streamline, then by the Newton's second law of motion, we have

$$\rho\alpha\,\delta s.\frac{Dq}{Dt}=\rho\alpha\,\delta s.S+p\alpha-\left(p+\frac{\partial p}{\partial s}\delta s\right)\alpha$$



$$\Rightarrow \frac{Dq}{Dt} = S - \frac{1}{\rho}\frac{\partial p}{\partial s}$$

Fig. 6.1

$$\Rightarrow \frac{\partial q}{\partial t} + q\frac{\partial q}{\partial s} = S - \frac{1}{\rho}\frac{\partial p}{\partial s} \qquad (6.16)$$

If the motion be steady $\frac{\partial q}{\partial t} = 0,$ and if the external forces have a potential function V such that $S = -\frac{\partial V}{\partial s}$, Eq. (6.16) reduces to

$$\frac{1}{2}\frac{\partial q^2}{\partial s} + \frac{\partial V}{\partial s} + \frac{1}{\rho}\frac{\partial p}{\partial s} = 0 \qquad (6.17)$$

If ρ is a function of p, integration of Eq. (6.17) along the streamline AB yields

$$\frac{1}{2}q^2 + V + \int \frac{dp}{\rho} = C, \qquad (6.18)$$

where C is a constant, whose value depends on the particular chosen streamline.

Case (i) If the fluid be homogeneous and incompressible, $\rho =$ constant and hence Eq. (6.18) reduces to

$$\frac{q^2}{2} + V + \frac{p}{\rho} = C. \qquad (6.19)$$

Case (ii) Let S be a gravitational force per unit mass. Let δh be the vertical distance between C and D. Then, we have

$$S = -g\frac{\partial h}{\partial s} = -\frac{\partial}{\partial s}(gh), \ as\ V = gh$$

Hence, if the fluid is incompressible, Eq. (6.18) reduces to

$$\frac{q^2}{2} + gh + \frac{p}{\rho} = C. \qquad (6.20)$$

It is also termed as "**Bernoulli's theorem**".

Example 1: A stream is rushing from a boiler through a conical pipe, the diameter of the ends of which are D and d; if V and v be the corresponding velocities of the stream and if the motion is supposed to be that of the divergence from the vertex of the cone, prove that

$$\frac{v}{V} = \left(\frac{D^2}{d^2}\right) e^{\left(v^2 - V^2\right)/2k}$$

where k is the pressure divided by the density and supposed constant.

Solution. Let AB and $A'B'$ be the ends of the conical pipe such that $A'B' = d$ and $AB = D$. Let ρ_1 and ρ_2 be densities of the stream at $A'B'$ and AB. By principle of conservation of mass, the mass of the stream that enters the end AB and leave at the end $A'B'$ must be the same. Hence, the equation of continuity is

$$\pi\left(\frac{d}{2}\right)^2 v\rho_1 = \pi\left(\frac{D}{2}\right)^2 V\rho_2$$

So that $$\frac{v}{V} = \frac{D^2}{d^2} \times \frac{\rho_2}{\rho_1} \qquad (1)$$

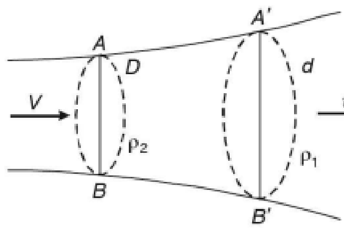


Fig. 6.2

By Bernoulli's theorem (in the absence of external forces like gravity), we have

$$\int \frac{dp}{\rho} + \frac{1}{2}q^2 = C \qquad (2)$$

Given that $\frac{p}{\rho} = k$ so that $dp = k\, d\rho$

(3)

Hence, Eq. (2) reduces to $k\int \frac{d\rho}{\rho} + \frac{1}{2}q^2 = C,$ using Eq. (3)

Integrating, $k \log \rho + \frac{q^2}{2} = C,$ C being an arbitrary constant.

(4)

When $q = v, \rho = \rho_1$ *and* *when* $q = V, \rho = \rho_{2.}$ Hence, Eq. (4) yields

$$k \log \rho_1 + \frac{v^2}{2} = C \;\; and \;\; k \log \rho_2 + \frac{V^2}{2} = C$$

Subtracting, $k\left(\log \rho_2 - \log \rho_1\right) + \frac{\left(V^2 - v^2\right)}{2} = 0$

$$\Rightarrow \log\left(\frac{\rho_2}{\rho_1}\right) = \frac{\left(v^2 - V^2\right)}{2k} \quad or \quad \frac{\rho_2}{\rho_1} = e^{\frac{\left(v^2 - V^2\right)}{2k}} \qquad (5)$$

Using Eq. (5), Eq. (1) reduces to

$$\frac{v}{V} = \left(\frac{D^2}{d^2}\right) \times e^{\left(v^2 - V^2\right)/2k}.$$

Example 2: A quantity of liquid occupies the length $2l$ of a straight tube of uniform small bore under the action of a force to a point in the tube varying as a distance from that point. Determine the pressure at any point.

OR

A quantity of liquid occupies the length $2l$ of a straight tube of uniform bore under the action of a force which is equal to μx to a point O in the tube, where x is the distance from O. Find the motion and show that if z be the distance of the nearer free surface from O, pressure at any point is given by $\frac{p}{\rho} = -\frac{1}{2}\mu\left(x^2 - z^2\right) + \mu\left(x - z\right)\left(z + l\right)$.

Solution. Let p be the pressure and u the velocity at a distance x from the fixed point O; and let z be the distance of the nearer free surface from O. Then, the equation of continuity is

$$\frac{\partial u}{\partial x} = 0 \qquad (1)$$

Let μx be the external force at a distance x which acts towards O. Then, the equation of motion

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} \quad \text{reduces to} \quad \frac{\partial u}{\partial t} = -\mu X - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad (2)$$

Integrating Eq. (2) with respect to 'x', we get

$$x\frac{\partial u}{\partial t} = -\frac{1}{2}\mu x^2 - \frac{p}{\rho} + C, C \text{ being an arbitrary constant} \qquad (3)$$

But $p = 0$ when $x = z$ and $x = z + 2l$. So, Eq. (3) gives

$$z\frac{\partial u}{\partial t} = -\frac{1}{2}\mu z^2 + C \qquad (4)$$

$$\left(z + 2l\right)\frac{\partial u}{\partial t} = -\frac{1}{2}\mu\left(z + 2l\right)^2 + C \qquad (5)$$

Subtracting Eq. (4) from Eq. (5), we get

$$2l\frac{\partial u}{\partial t} = -\frac{1}{2}\mu\left[\left(z+2l\right)^2 - z^2\right] + C \quad or \quad \frac{\partial u}{\partial t} = -\mu\left(z+l\right) \tag{6}$$

$$\Rightarrow \frac{d^2 z}{dt^2} = -\mu\left(z+l\right) \quad \left[\because u = \frac{dz}{dt}\right] \tag{7}$$

Putting $z + l = y$ so that $z = y - l$. Then, Eq. (7) gives

$$\frac{d^2 y}{dt^2} + \mu y = 0$$

whose solution is $y = A\cos\left(t\sqrt{\mu} + B\right)$, $A \; and \; B$ being arbitrary constants.

Since $y = z + 1$, it yields $z = A\cos\left(t\sqrt{\mu} + B\right) - l$ (8)

in which $A \; and \; B$ may be determined from the knowledge of initial position and velocity.

We now determine pressure. From Eq. (4), we get

$$C = z\frac{\partial u}{\partial t} + \frac{1}{2}\mu z^2$$

Putting this value of C in Eq. (3), we get

$$\frac{p}{\rho} = -\frac{1}{2}\mu\left(x^2 - z^2\right) - \left(x - z\right)\frac{\partial u}{\partial t} \quad or \quad \frac{p}{\rho} = -\frac{1}{2}\mu\left(x^2 - z^2\right) + \mu\left(x - z\right)\left(z + l\right),$$

using Eq. (6) which gives the pressure at any point.

Example 3: A jet of water 8 cm. in diameter impinges on a plate held normal to its axis. For a velocity of 4 m/sec., what force will keep the plate in equilibrium?

Solution. Diameter of jet = d = 8 cm. = 0.08 m.

$\therefore$ Area of cross-section of the jet = $S = \left(\frac{\pi}{4}\right) \times d^2 = \left(\frac{\pi}{4}\right) \times (0.08)^2 \, m^2$

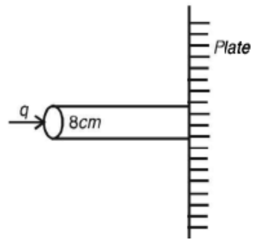


Fig. 6.3

$q =$ velocity of jet = 4 m/sec.

$w =$ weight per unit cubic meter of water = 10^3 kg/m^3

$F =$ Force acting on the jet.

Now, force on the plate = charge in momentum

$$\Rightarrow F = \frac{w(Sq)q}{g} = \frac{wSq^2}{g} = \frac{1000 \times (\pi/4) \times (0.08)^2 \times (4)^2}{9.81} = 32.8\,kg.$$

6.5 APPLICATIONS OF BERNOULLI'S EQUATION

Bernoulli's equation is of fundamental importance in the fluid dynamics, especially in hydraulics. It is employed to handle some complicated situations of fluid flow problems in a simple manner. We now discuss some practical applications of the Bernoulli's equation. In each case the fluid will be assumed inviscid and incompressible.

6.5.1 PITOT TUBE

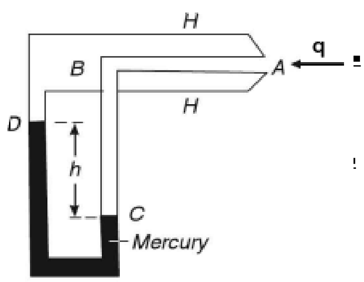


Fig. 6.4

A Pitot tube is an instrument to measure the velocity of flow at the required point in a pipe or a stream. Suppose we wish to determine the velocity q of a stream of water. The inner tube BA is kept so as to face the direction of the flow as shown in figure. The outer tube of the Pitot tube has holes such as H. If p is the pressure in the stream where the fluid velocity is q then p

is also the pressure on the inside and outside of the hole and therefore p is also the pressure at the meniscus D of the mercury in the $U-$tube (manometer). Let the stream enter the tube AB and let it be brought to rest at meniscus C. C is called a stagnation point. Let p_0 be pressure at C. Applying the Bernoulli's equation to the streamline passing through A and C, we have

$$\frac{p}{\rho}+\frac{1}{2}q^2=\frac{p_0}{\rho} \Rightarrow q=\sqrt{\left\{\frac{2(p_0-p)}{\rho}\right\}}, \tag{6.21}$$

where ρ is the density of the water.

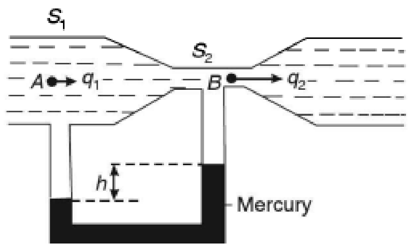
Let h be the difference in level of the mercury in the $U-$tube and let σ be the density of the mercury. Then we have

$$p_0-p=\sigma gh \tag{6.22}$$

Using (6.22), (6.21) reduces to

$$q=\left(2\sigma gh/\rho\right)^{1/2} \tag{6.23}$$

which determines the fluid velocity at a point in the flow region.



6.5.2 *VENTURI METER (OR TUBE)*

A venturi meter is an instrument to measure the fluid velocity in pipes. The flow rate of a fluid in conduit and the discharge of a fluid flowing in a pipe may also be measured. The venturi meter is made up of a constant cross-section S_1 tapering to a section of small cross-section S_2 (also known as throat) and then gradually expanding to the original cross-section. A $U-$tube serving as a mercury manometer is attached to connect the broad and narrow sections at A and B.

Fig. 6.5

Let q_1, q_2 be the fluid velocities at A, B and p_1, p_2 the pressures. Then by the equation of continuity, we have

$$q_1 S_1 = q_2 S_2 \quad \Rightarrow \quad q_2 = (q_1 S_1)/S_2 \tag{6.24}$$

Applying the Bernoulli's equation to the central streamline passing through A and B, we get

$$\frac{p_1}{\rho} + \frac{1}{2} q_1^2 = \frac{p_2}{\rho} + \frac{1}{2} q_2^2 \ , \tag{6.25}$$

where ρ is the density of the fluid. Eliminating q_2 from (6.24) and (6.25), we have

$$q_1 = \left\{ \frac{2(p_1 - p_2) S_2^2}{\rho\left(S_1^2 - S_2^2\right)} \right\}^{1/2} \tag{6.26}$$

Let h be the difference in levels of the mercury in the $U-$tube and let σ be the density of the mercury. Then, we have $$p_1 - p_2 = \sigma g h \tag{6.27}$$

Using (6.27), (6.26) reduces to $$q_1 = \left\{ \frac{2\sigma g h S_2^2}{\rho\left(S_1^2 - S_2^2\right)} \right\}^{1/2} \tag{6.28}$$

Let Q be the flow rate of the fluid flowing through the broad section at A. Then

$$q_1 = \rho q_1 S_1 = \rho S_1 \left\{ \frac{2\sigma g h S_2^2}{\rho\left(S_1^2 - S_2^2\right)} \right\}^{1/2} \tag{6.29}$$

Remarks: Let the venturi meter be kept inclined at a certain angle to the horizon. With reference to a fixed horizontal line, let vertical heights of A and B be h_1 and $h_2 \left(h_2 > h_1\right)$ and let $h_2 - h_1 = \Delta h$. then equation (6.25) modifies in the following form:

$$\frac{p_1}{\rho}+\frac{1}{2}q_1^2+gh_1=\frac{p_2}{\rho}+\frac{1}{2}q_2^2+gh_2 \qquad (6.30)$$

Eliminating q_1 from (6.24) and (6.30), we get

$$q_2=\left\{\frac{2\left[\left(p_1-p_2\right)/\rho-g\left(h_2-h_1\right)\right]}{\left(1-\left(S_2^2/S_1^2\right)\right)}\right\}^{1/2}$$

And hence the flow rate at either sections is given by

$$Q=S_2q_2=S_2\left\{\frac{2\left[\left(\sigma gh\right)/\rho-g\Delta h\right]}{\left(1-\left(S_2^2/S_1^2\right)\right)}\right\}^{1/2} \qquad (6.31)$$

Let C be the coefficient of venturi meter (or the coefficient of discharge). Let Q be the discharge through the venturi meter. Then we know that

$$Q=CS_2q_2=CS_2\left\{\frac{2\left[\left(\sigma gh/\rho\right)-g\Delta h\right]}{\left(1-\left(S_2^2/S_1^2\right)\right)}\right\}^{1/2} \qquad (6.32)$$

If $\Delta h=0$ (i.e., the venturi meter is horizontal), then (6.32) reduces to

$$Q=\frac{CS_1S_2}{\sqrt{S_1^2-S_2^2}}\sqrt{\frac{\sigma}{\rho}}\sqrt{2gh}. \qquad (6.33)$$

6.5.3 *FLOW FROM A TANK THROUGH A SMALL ORIFICE*

Consider a tank containing a liquid. Let the tank be sealed except for a small orifice near the base. We wish to determine the velocity of efflux from the tank when the orifice is opened. Let S_1 and S_2 be the areas of cross-section of the tank and the orifice respectively.

Now the water will move out steadily in the form of a smooth jet. Let the line connecting point 1 on the liquid surface with the point 2 in the jet represents a streamline of the flow. Then, the Bernoulli's theorem yields

$$\frac{p_1}{\rho}+\frac{1}{2}q_1^2+gh_1=\frac{p_2}{\rho}+\frac{1}{2}q_2^2+gh_2 \qquad (6.34)$$

But from figure $$h_1-h_2=h \qquad (6.35)$$

Now, from the equation of continuity, we have

$$q_1S_1=q_2S_2 \quad \Rightarrow \quad q_1=(S_2/S_1)\times q_2 \qquad (6.36)$$

Using (6.35) and (6.36), (6.34) reduces to

$$\frac{1}{2}q_2^2-\frac{1}{2}\frac{S_2^2}{S_1^2}q_2^2=g(h_1-h_2)+\frac{1}{\rho}(p_1-p_2)$$

$$\Rightarrow \frac{1}{2}q_2^2\times\left(1-\frac{S_2^2}{S_1^2}\right)=gh+\frac{1}{\rho}(p_1-p_2)$$

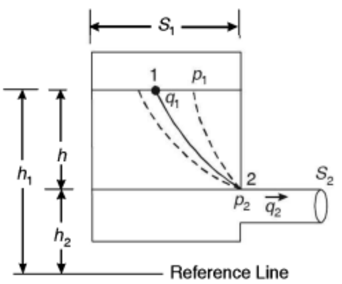


Fig. 6.6

$$\Rightarrow q_2=\sqrt{\frac{2}{\left(1-\frac{S_2^2}{S_1^2}\right)}\left(\frac{p_1-p_2}{\rho}+gh\right)}$$

(6.37)

which gives the desired velocity of efflux from the tank through the orifice.

We now discuss two special cases of (6.37):

Case (i) Suppose the tank is vented to the atmosphere or has an open surface, so that $p_1 = p_2$. Further, let $S_2 << S_1$. then, (6.37) reduces to

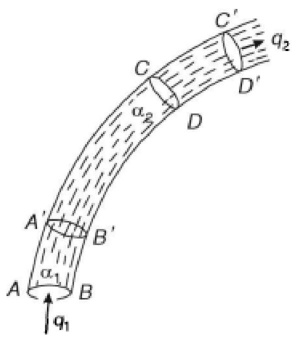
$$q_2 = \sqrt{2gh}. \tag{6.38}$$

Hence, the velocity of efflux from the vented tank is equal to that of a rigid body falling freely from a height h.

The above result is known as **Torricelli's theorem**.

Case (ii) Let $S_2 << S_1$ and let $\dfrac{(p_1 - p_2)}{\rho} >> gh$. Then (6.37) reduces to

$$q_2 = \sqrt{\frac{2(p_1 - p_2)}{\rho}}.$$



6.6 *EULER'S MOMENTUM THEOREM*

Consider steady motion of a non-viscous liquid contained between, AB and CD of the filament at a given time t. the surrounding fluid will produce a force on the walls and ends of the filament. By Newton's second law of motion, the net force will be equal to the rate of change of momentum of the fluid in the filament $ABCD$ at time t. At time $t + \delta t$, let the new position of the fluid be $A'B'C'D'$. them notice that the momentum of the given fluid has increased by the momentum of the fluid between CD and $C'D'$ and has decreased by the momentum of the fluid between AB and $A'B'$.

$\therefore$ Gain of momentum at $CD = (\rho \alpha_2 q_2 \delta t) q_2$

and loss of momentum at $AB = (\rho\alpha_1 q_1 \delta t) q_1$

where q_1 and q_2 are the velocities at AB and CD respectively.

Hence, the net gain = $\rho\delta t\left(\alpha_2 q_2^2 - \alpha_1 q_1^2\right)$

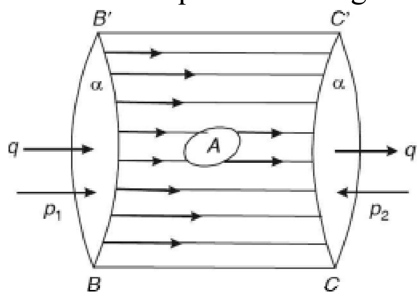
Fig. 6.7

or the rate of gain = $\rho\left(\alpha_2 q_2^2 - \alpha_1 q_1^2\right)$.

This gives the resultant force due to the pressure of the surrounding liquid on the walls and ends of the filament. This result is known as "**Euler's momentum theorem**".

6.7 *D'ALEMBERT'S PARADOX*

Consider a long straight channel of uniform cross section in which a liquid is flowing with a uniform speed q. Let the ends of the tube be bounded by equal cross-sectional area α. If an obstacle A is placed in the middle of the channel, the flow in the immediate neighbourhood of A will be disturbed whereas the flow at a great distance either up-stream or down-stream will remain undisturbed. Suppose F is the force required to hold the obstacle to rest, in the direction of uniform flow.



Let BB' and CC' be two sections at a great distance from A and let the fluid between these sections be split up into stream filaments. Since the outer filaments are bounded by the walls of the channel, the thrust components are normal to the direction of flow. Moreover, the obstacle A acts on those filaments which are in contact with it by a force $-F$.

By Euler's momentum theorem the resultant of all the thrusts on the fluid is $\rho\alpha q^2 - \rho\alpha q^2$.

Let p_1 and p_2 be the pressures on BB' and CC' respectively. Then, Bernoulli's theorem gives

$$\frac{p_1}{\rho}+\frac{1}{2}q^2=C=\frac{p_2}{\rho}+\frac{1}{2}q^2$$ so that

Fig. 6.8

$p_1 = p_2$

Now, the thrust due to pressure p_1 and p_2 is $p_1\alpha - p_2\alpha$.

Thus, the equation of motion becomes

$$p_1\alpha - p_2\alpha - F = p\alpha q^2 - p\alpha q^2$$ so that $F=0, \quad as \quad p_1 = p_2$

Let the diameter of the channel increase indefinitely. Then the above problem reduces to that of an obstacle immersed in an infinite uniform stream. As before, again the resultant force exerted by the liquid on the obstacle is zero.

Now let us superimpose a velocity u in the opposite direction on the entire system (the body A and the liquid). Then, the body A can be thought as moving with uniform velocity u and the liquid at great distance is reduced to rest.

Thus, a body moving with uniform velocity through an infinite liquid, otherwise at rest, will experience no resistance at all. This result is known as **D'Alembert's paradox**.

6.8 *SUMMARY*

In this unit we studied:

(i) D'Alembert's paradox.

(ii) Euler's Equation of Motion under Conservative Body Forces.

(iii) Euler's momentum theorem.

6.9 *REFERENCES AND SUGGESTED READINGS*

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(ii) Frank M. White (2011), *Fluid Mechanics*, McGraw Hill.

(iii) John Cimbala and Yunus A Çengel (2019), *Fluid Mechanics: Fundamentals and Applications*, McGraw Hill.

(iv) P.K. Kundu, I.M. Cohen & D.R. Dowling (2015), Fluid Mechanics, Academic Press; 6th edition.

(v) F.M. White & H. Xue (2022), Fluid Mechanics, McGraw Hill; Standard Edition.

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6.10 *TERMINAL QUESTIONS*

1. If the motion is steady, velocity potential does not exist and V be the potential function from which the external forces are derivable, then Bernoulli's theorem is

(i) $-\frac{\partial \phi}{\partial t}+\frac{1}{2}q^2+V+\int\frac{dp}{\rho}=C$

(ii) $\int\frac{dp}{\rho}+\frac{1}{2}q^2+V=C$

(iii) $\frac{p}{\rho}+\frac{q^2}{2}+V=C$

(iv) None of these.

2. The equation $\frac{q^2}{2}+\Omega+\frac{p}{\rho}=const.$ is known as

(i) Navier equation

(ii) Stokes equation

(iii) Bernoulli equation

(iv) Euler equation

3. For a perfect incompressible liquid, flowing in a continuous stream, the total energy of a particle remains the same, while the particle moves from one point to another. This statement is called:

   (i) Continuity equation

   (ii) Archimedes principle

   (iii) Pascal's law

   (iv) Bernoulli's equation

4. The Bernoulli's equation is based on the assumption that:

   (i) There is no loss of energy of the liquid flowing.

   (ii) The velocity of flow is uniform across any cross-section.

   (iii) No force except gravity acts on the fluid.

   (iv) All of the above.

5. The Euler's equation for the motion of liquids is based upon the assumption that:

   (i) The fluid is non-viscous, homogeneous and incompressible.

   (ii) The velocity of flow is uniform over the section.

   (iii) The flow is continuous, steady and along the stream line.

   (iv) All of the above.

6. The Bernoulli's equation for unsteady and irrotational motion is given by:

(i) $-\frac{\partial \phi}{\partial t}+\frac{q^2}{2}+V+\frac{p}{\rho}=F(t)$

(ii) $-\frac{\partial \phi}{\partial t}+\frac{q^2}{2}+V=F(t)$

(iii) $-\frac{\partial \phi}{\partial t}-\frac{q^2}{2}+V-\frac{p}{\rho}=F(t)$

(iv) $\frac{q^2}{2}+V+\frac{p}{\rho}=F(t)$

7. A horizontal pipe gradually reduces in diameter from 24 in. to 12 in. Determine the total longitudinal thrust exerted on the pipe if the pressure at the larger end is 50 lbf/in^2 and the velocity of the water is 8 ft./sec.

8. Calculate the force exerted by a jet of water 3/4 in. in diameter which strikes a flat plate at an angle of 30° to the normal of the plate with a velocity of 30 ft/sec if (a) the plate is stationary, (b) the plate is moving in the direction of the jet with a velocity of 10 ft/sec.

9. Briefly explain the application of Bernoulli's theorem.

10. State and prove D'Alembert's paradox.

11. A stream in a horizontal pipe, after passing a contraction in the pipe at which its sectional area is A delivered at atmospheric pressure at a place, where the sectional area is B . show that if a side tube is connected with the pipe at the former place, water will be sucked up through it into the pipe from a reservoir at a depth $\left(s^2/2g\right)\times\left(1/A^2-1/B^2\right)$ below the pipe, s being the delivery per second.

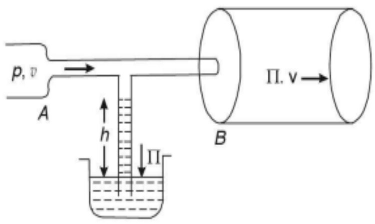


Fig. 6.9

12. Air, obeying Boyle's law, is in motion in a uniform tube of a small section prove that if ρ be the density and v the velocity at a distance x from a fixed point at time t, then

$$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2}{\partial x^2}\left\{\rho\left(v^2 + k\right)\right\}, \textit{ where } k = \frac{p}{\rho}.$$

13. If the body force $\vec{F}$ form a conservative system, density ρ is a function of p only and the flow is steady, prove that $\Omega + P + \vec{q}^2/2$ is constant along every streamline and vortex line, where $\vec{F} = -\nabla\Omega, \quad P = \int\left(\frac{1}{\rho}\right)dp \textit{ and } \vec{q} \textit{ is velocity}.$

6.11 ANSWERS

1. (ii)
2. (iii)
3. (iv)
4. (iv)
5. (iv)
6. (i)
7. 95040π
8. (a) 4.63 Ibf. (b) 3.09 Ibf.

Course Name: FLUID MECHANICS
Course Code: MAT604

BLOCK-III
TWO DIMENSIONAL FLOW

UNIT 7: *TWO DIMENSIONAL FLOWS*

CONTENTS:

7.1 *INTRODUCTION*

Two-dimensional flows in fluid dynamics involve the study of fluid motion confined to a plane described by velocity components dependent on two spatial coordinates. Key concepts include the stream function, which defines streamlines, and the potential function for irrotational flows. Vorticity measures local fluid rotation. Types of two-dimensional flows include potential flow, vortex flow, source/sink flow, and uniform flow. These concepts are crucial for applications in aerodynamics, hydrodynamics, and environmental science, helping to analyze and predict fluid behavior in various practical scenarios.

7.2 *OBJECTIVES*

After completion of this unit learners will be able to:

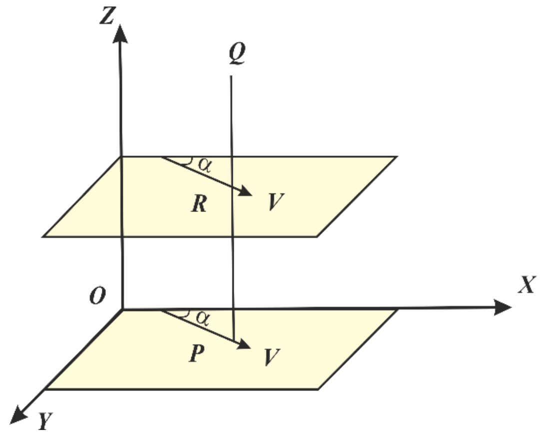
(i). Understand fluid flow in two dimensions.

(ii). Use cylindrical polar coordinates to study fluid flow.

7.3 *MEANING OF TWO –DIMENSIONAL FLOW*

Let a fluid move in such a way that at any given instant the flow pattern in a certain plane (say XOY) is the same as that in all other parallel planes within the fluid. Then the fluid is said to have two-dimensional motion. If (x, y, z) are coordinates of any point in the fluid, then all physical quantities (velocity, density, pressure etc.) associated with the fluid are independent of z. Thus u, are functions of x, y and t and $w = 0$ for such a motion.

To make the concept of two-dimensional motion clearer, suppose the plane under consideration be xy-plane. Let P be an arbitrary point on that plane. Draw a straight line PQ parallel to OZ (or perpendicular to the xy-plane). Then all points on the line PQ are said to correspond to P. Draw a plane (in the fluid) parallel to the xy-plane and meeting PQ in R. Then, if the velocity at P is V in the xy plane in a direction making an angle α with OX, the velocity at R is also V in magnitude and parallel in direction to the velocity at P as shown in the figure. It follows that the velocity at corresponding points is a function of x, y and the time t, but not of z.



To maintain physical reality, we assume that the fluid in two-dimensional motion is confined between two planes parallel to the plane of motion and at a unit distance apart. The reference plane of motion is taken parallel to and midway between the assumed fixed planes. Thus, while studying the flow of a fluid past a cylinder in a two-dimensional motion in planes perpendicular to the axis of the cylinder, it is useful to restrict attention to a unit length of cylinder confined between the said planes in place of worrying over the cylinder of infinite length.

Suppose we are dealing with a two-dimensional motion in xy plane. Then by flow across a curve in this plane, we mean the flow across unit length of a cylinder whose trace on the plane xy is the curve under consideration, the generators of the cylinder being parallel to the z-axis. By a point in a flow, we mean a line through that point parallel to z-axis.

7.4 *EQUATION OF CONTINUITY FOR TWO-DIMENSIONAL FLOW*

Consider a rectangular prism of fluid of an elementary cross-section ABCD (Fig 7.1). Let its size be δx by δy in the X-Y plane and of unit length perpendicular to the plane. At the centroid (X, Y) of ABCD, let the fluid density be ρ, u the X-component and v the Y-component of velocity.

Mass rate of flow across $EF = \rho u \delta y$

and mass rate of flow across GH $= \rho v \delta x$.

The rate of change of any quantity with respect to distance in X-direction is mathematically expressed as $\partial / \partial x$ of that quantity. Likewise, $\partial / \partial y$ of any quantity represents the rate of change of the quantity with respect to distance in Y-direction.

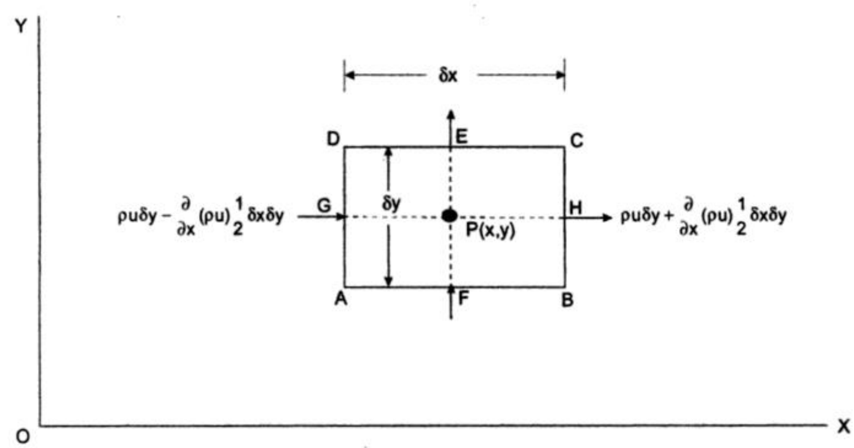


Fig 7.1: Flow through a Two-dimensional Rectangular Fluid Element

∴ Mass rate of flow entering the side

$$AB = m_1 = \rho u\delta y - \frac{\partial}{\partial x}(\rho u)\frac{1}{2}\delta x\delta y,$$

and mass rate of flow leaving the side

$$CD = m_2 = \rho u\delta y + \frac{\partial}{\partial x}(\rho u)\frac{1}{2}\delta x\delta y.$$

In unit time, therefore, the gain in mass in X-direction is equal to

$$m_x = m_1 - m_2 = -\frac{\partial}{\partial x}(\rho u)\delta x\delta y$$

By a similar analysis in y-direction, it can be shown that the gain in mass per unit time in Y-direction is equal to

$$m_y = -\frac{\partial}{\partial x}(\rho u)\delta x\delta y$$

The total gain in mass per unit time is

$$m = m_x + m_y = -\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v)\right]\delta x\delta y$$

According to the principle of conservation of mass, m should be equal to the time rate of increase of mass within the element, viz., $\frac{\partial}{\partial \mathrm{t}}(\rho\delta x\delta y)$

$$\therefore -\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v)\right]\delta x\delta y = \frac{\partial}{\partial t}(\rho\delta x\delta y)$$

$$\text{or } \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \qquad (1)$$

Which is the continuity equation for two-dimensional unsteady compressible flow. If the flow is steady, there would be no change with respect to time and the equation reduces to

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \qquad (2)$$

If the flow is incompressible, the density is constant and hence ρ can be taken outside the differential. further (2) reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (2)$$

which is the continuity equation for two-dimensional steady incompressible flow.

7.4.1 *EXAMPLES BASED ON EQUATION OF CONTINUITY*

Example 1. Consider a velocity field $u = x^2$ and $v = -2xy$. Determine if this field satisfies the continuity equation.

Solution. Calculate the partial derivatives:

$$\frac{\partial u}{\partial x} = \frac{\partial (x^2)}{\partial x} = 2x$$
$$\frac{\partial v}{\partial y} = \frac{\partial (-2xy)}{\partial y} = -2x$$

Now, add these derivatives:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2x + (-2x) = 0$$

The given velocity field satisfies the continuity equation.

Example 2. Consider a velocity field $u = 2x - y^2$ and $v = 4y$. Verify if it satisfies the continuity equation.

Solution. Calculate the partial derivatives:

$$\frac{\partial u}{\partial x} = \frac{\partial (2x - y^2)}{\partial x} = 2$$
$$\frac{\partial v}{\partial y} = \frac{\partial (4y)}{\partial y} = 4$$

Now, add these derivatives:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2 + 4 = 6$$

Since the sum is not zero, the given velocity field does not satisfy the continuity equation.

Example 3. Examine whether the velocity field: $V = 2ax(3y^2 - x^2)i + 2ay\,(3x^2 - y^2)j$ represents a possible two-dimensional incompressible fluid flow.

Solution. From the given velocity field, it is clear that

$$u = 2ax(3y^2 - x^2) \text{ and } v = 2ay(3x^2 - y^2)$$

A two - dimensional incompressible flow must satisfy the continuity equation: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}[2ax(3y^2 - x^2)]$$
$$= \frac{\partial}{\partial x}(6ay^2 - 2ax^3) = 6ay^2 - 6ax^2 = 6a(y^2 - x^2)$$
$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}[2ay(3x^2 - xy^2)]$$
$$= \frac{\partial}{\partial y}(6ax^2y - 2ay^3)$$
$$= 6ax^2 - 6ay2 = 6a(x^2 - y^2)$$
$$\therefore \ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 6a(y^2 - x^2) + 6a(x^2 - y^2) = 0$$

The continuity equation is satisfied. Hence the given velocity field represents a possible two-dimensional incompressible flow.

Example 4. The velocity distribution for the flow of an incompressible fluid is given by $u = 3 - 2x,$ and $v = 4 + 2y$. Show that this satisfies the requirements of the continuity equation.

Solution. For two-dimensional flow of an incompressible fluid, the continuity equation simplifies to $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$

$$\frac{\partial u}{\partial x}=-1,\frac{\partial v}{\partial y}=2,$$

Therefore,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=-2+2=0$$

Which satisfies the requirement for continuity. Hence two-dimensional flow of an incompressible fluid.

7.5 *USE OF CYLINDRICAL POLAR COORDINATES*

For an incompressible irrotational flow of uniform density, the equation of continuity $\Delta^2\phi=0$ for the velocity potential $\phi(r,\theta,z)$ in cylindrical polar co-ordinates (r,θ,z) is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\partial^2\phi}{\partial z^2}=O \qquad (1)$$

If the flow is two dimensional and the co-ordinate axes are to so choose that all physical quantities associated with the fluid are independent of z then $\phi=\phi(r,\theta)$

∴ (1) becomes,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}=O \qquad (2)$$

Let $\phi(r,\theta)=-f(r)g(\theta)$ be the solution of Equation (2) for separation of variables.

Thus, we get

$$g(\theta)\frac{1}{r}\frac{d}{dr}[rf'(r)] + \frac{1}{r^2}f(r)g''(\theta) = 0 \tag{3}$$

$$\frac{r\frac{d}{dr}[rf'(r)]}{f(r)} = -\frac{g''(\theta)}{g(\theta)} \tag{4}$$

Thus, L.H.S of Equation (4) is a function of r only and RHS is a function of θ only.

As r, θ are independent variables. So, each side of Equation (4) is a constant say λ.

$$\frac{r^2 f''(r) + rf'(r)}{f(r)} = -\frac{g''(\theta)}{g(\theta)} = \lambda$$

i.e., $r^2 f''(r) + rf'(r) - \lambda f(r) = 0$ (5)

$$g''(\theta) + \lambda g(\theta) = 0 \tag{6}$$

Equation (6) has periodic solution when $\lambda > 0$ normally the physical problem requires that $g(\theta + 2\pi) = g(\theta)$and this is satisfied when $\lambda = n^2$ for $n = 1,2,3,\cdots$

The basic solution of Equation (6) are

$$g''(\theta) + \lambda g(\theta) = 0 \tag{7}$$

Equation (5) is of Euler homogeneous type, and it is reduced to a linear different equation of constant co-efficient by putting

$$r = e^t$$
$$t = \log r$$
$$\frac{dt}{dr} = \frac{1}{r}$$
$$f'(r) = \frac{df}{dr} = \frac{df}{dt} \times \frac{dt}{dr} = \frac{1}{r} \times \frac{df}{dt}$$
$$f''(r) = \frac{d^2 f}{dr^2} = \frac{d}{dr}\left(\frac{1}{r} \times \frac{df}{dt}\right) = \frac{1}{r}\frac{d}{dr}\left(\frac{df}{dt}\right) - \frac{1}{r^2}\frac{df}{dt}$$
$$= \frac{1}{r}\left[\frac{d}{dt}\left(\frac{df}{dt}\right)\frac{dt}{dr}\right] - \frac{1}{r^2}\frac{df}{dt}$$
$$= \frac{1}{r^2}\frac{d^2 f}{dt^2} - \frac{1}{r^2}\frac{df}{dt}$$
$$r^2 f''(r) = \frac{d^2 f}{dt^2} - \frac{df}{dt}$$

Equation (5) reduces to

$$\frac{d^2 f}{dt^2} - \frac{df}{dt} + \frac{df}{dt} - n^2 f = 0$$
$$\frac{d^2 f}{dt^2} - n^2 f = 0$$

Solution is $f = e^{\pm nt} = (e^t)^{\pm n} = r^{\pm n}$

$$c_3 r^n + c_4 r^{-n} \tag{8}$$

A special solution of Equation (2) is obtained by Equation (7) and (8) as

$$\phi(r,\theta) = -f(r)g(\theta)$$
$$\phi(r,\theta) = -(c_3 r^n + c_4 r^{-n})(c_1 cos\, n\theta + c_2 sin\, n) \tag{9}$$

The most general solution is

$$\phi(r,\theta) = -\sum_{n=1}^{\infty} (A_n r^n + B_n r^{-n})(C_n \cos n\theta + D_n \sin n\theta) \tag{10}$$

Case,

For $n = 0$ we have,

$$f = k_1 + k_2 t = k_1 + k_2 log\, r$$
$$g = k_3 + k_4 \theta$$

So, another solution of Equation (2) is

$$\phi(r,\theta) = -(k_1 + k_2 log\, r)(k_3 + k_4\theta)$$

For $n = 1$

$$\phi = -r\cos\theta \qquad \phi = -r\sin\theta$$
$$\phi = -r^{-1}\cos\theta \quad \phi = -r^{-1}\sin\theta$$

Discuss the uniform flow part as infinitely long circular cylinder.

Let P be a point with cylindrical polar co-ordinates (r,θ,z) in the flow region of an unbounded.

Incompressible fluid of uniform density moving irrotationally with uniform velocity $-Ui$ at infinity past the fixed solid cylinder $r \leq a$.

When the cylinder $r = a$ is introduced, it will produce a perturbation which is such as to satisfy Laplace equation and to become vanishingly small for large r.

This suggests taking the velocity potential for $r \leq a, 0 \leq 0 \leq 2\pi$ in the form

$$\phi(r,\theta) = Urcos\,\theta - Ar^{-1}cos\,\theta \tag{11}$$

Where the velocity potential of the uniform stream is $Ux = Urcos\,\theta$
and due to perturbation it is $-Ar^{-1}cos\,\theta$ which tends to zero as $r \to \infty$ and gives rise to a velocity pattern which is symmetrical about $\theta = 0, \pi$ (the term $r^{-1}sin\,\theta$ is not there since it does not give symmetric flow)

As there is no flow across $r = a_1$ so the boundary condition on the surface is

$$\frac{\partial \phi}{\partial r} = 0 \ \ When \ r = a$$
$$\phi = Ur\cos\theta - Ar^{-1}\cos\theta$$
$$\frac{\partial \phi}{\partial r} = U\cos\theta + Ar^{-2}\cos\theta$$
$$When \ r = a, \frac{\partial \phi}{\partial r} = 0, 0 \leq \theta \leq 2\pi$$
$$0 = U + Aa^{-2}$$
$$0 = Ua^2 + A$$
$$A = Ua^2$$

Thus, velocity potential for an uniform flow part a fixed infinite cylinder is

$$\phi(r,\theta) = Ur\cos\theta + U\frac{a^2}{r}\cos\theta$$

$$= U\cos\theta\left(r + \frac{a^2}{r}\right) \rightarrow (3), r > a, 0 \leq \theta \leq 2\pi$$

From here, the cylindrical components of velocity are $(\vec{q} = \nabla\phi)$

$$q_r = \frac{-\partial \phi}{\partial r} = -U\cos\theta\left[1 - \frac{a^2}{r^2}\right]$$
$$q_\theta = \frac{-1}{r}\frac{\partial \phi}{\partial \theta} = \frac{-1}{r}U\sin\theta\left[r + \frac{a^2}{r}\right]$$

$$= U\sin\theta\left(1 + \frac{a^2}{r^2}\right)$$
$$q_x = \frac{-\hat{\partial}\phi}{\partial z} = 0$$

We note that as $r \rightarrow \infty, q_r = -U\cos\theta, q_\theta = U\sin\theta$ which are consistent with the velocity at infinity $-Ui$ of the uniform stream.

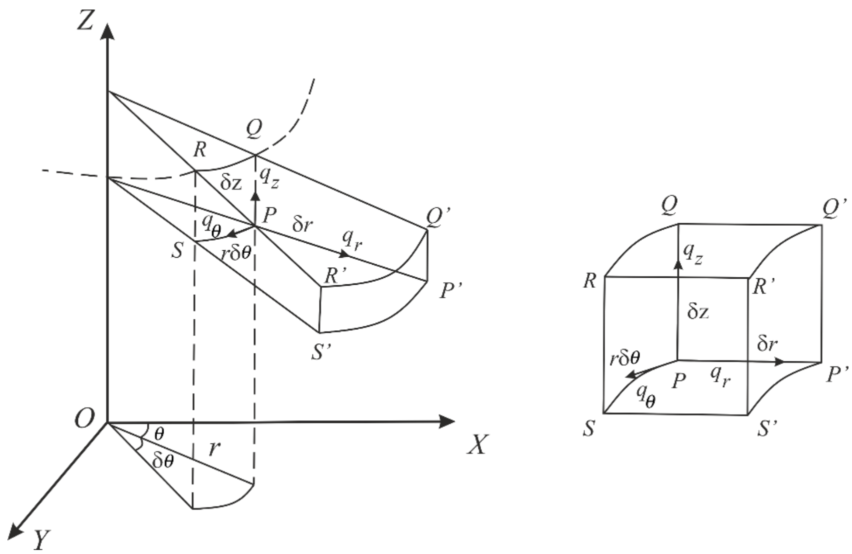
7.5.1 EXAMPLES BASED ON CYLINDRICAL POLAR COORDINATES

Example 5. Derive the equation of continuity in cylindrical coordinates**.**

Solution. Consider a fluid particle at P whose cylindrical coordinates are (r, θ, z),

where $r \geq 0, 0 \leq \theta \leq 2\pi, -\infty < z < \infty$.

Let $\rho(r, \theta, z, t)$ be the density of the fluid at P at any time t. With P as one corner construct a small curvilinear parallelepiped ($PQRS, P'Q'R'S'$) with its edges $SS' = \delta r$, arc $SP = r\delta\theta$ and $PQ = \delta z$. Let q_r, q_θ and q_z be the velocity components in the direction of the elements SS', arc SP and PQ respectively.



Then, we have

Mass of the fluid that passes in through the face $PSRQ$

$$= \rho \cdot r\delta\theta\delta z \cdot q_r \text{ per unit time} = f(r, \theta, z), \text{ say} \qquad (1)$$

$\therefore$ Mass of the fluid that passes out through the opposite face $P'S'R'Q'$

$$= f(r+\delta r,\theta,z) \text{ per unit time } = f(r,\theta,z) + \delta r\frac{\partial}{\partial r}f(r,\theta,z) + \cdots \qquad (2)$$

(expanding by Taylor's theorem)

∴ The net gain in mass per unit time within the chosen elementary parallelepiped ($PQRS, P'Q'R'S'$) due to flow through the faces $PSRQ$ and $P'S'R'Q'$ by using (1) and (2) = Mass that enters in through the face $PQRS$ - Mass that leaves through the face $P'Q'R'S'$

$$= f(r,\theta,z) - \left[f(r,\theta,z) + \delta r\cdot\frac{\partial}{\partial r}f(r,\theta,z) + \cdots\right]$$

$= -\delta r\cdot\frac{\partial}{\partial r}f(r,\theta,z)$, to the first order of approximation $= -\delta r\cdot\frac{\partial}{\partial r}(\rho r\delta\theta\delta z q_r)$, by equation (1)

$$= -\delta r\delta\theta\delta z\frac{\partial(\rho r q_r)}{\partial r} \qquad (3)$$

Similarly, the net gain in mass per unit time within the element due to flow through the faces $SRR'S'$ and $QPP'Q'$

$$= -\delta r\delta\theta\delta z\frac{\partial}{\partial\theta}(\rho q_\theta) \qquad (4)$$

and the net gain in mass per unit time within the element due to flow through the faces $PS'P'$ and $QRR'Q'$

$$= -\delta r\delta\theta\delta z\frac{\partial}{\partial z}(\rho r q_z) = -r\delta r\delta\theta\delta z\frac{\partial(\rho q_z)}{\partial z} \qquad (5)$$

∴ Total rate of mass flow into the chosen element

$$= -\delta r\delta\theta\delta z\left[\frac{\partial}{\partial r}(\rho r q_r) + \frac{\partial}{\partial\theta}(\rho q_\theta) + r\frac{\partial}{\partial z}(\rho q_z)\right] \qquad (6)$$

Again, the mass of the fluid within the element at time $t = \rho r\delta r\delta\theta\delta z$

∴ Total rate of mass increase within the element

$$= \frac{\partial}{\partial t}(\rho r\delta r\delta\theta\delta z) = r\delta r\delta\theta\delta z\frac{\partial\rho}{\partial t} \tag{7}$$

Suppose that the chosen region of the element of the fluid contains neither sources nor sinks. Then by the law of conservation of the fluid mass, the rate of increase of the mass of the fluid within the element must be equal to the rate of mass flowing into the element. Hence from (6) and (7), we have

$$r\delta r\delta\theta\delta z\frac{\partial\rho}{\partial t} = -\delta r\delta\theta\delta z\left[\frac{\partial}{\partial r}(\rho r q_r) + \frac{\partial}{\partial\theta}(\rho q_\theta) + r\frac{\partial}{\partial z}(\rho q_z)\right]$$

or $$\frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r q_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(\rho q_\theta) + \frac{1}{\partial z}(\rho q_z) = 0, \tag{8}$$

which is the desired equation of continuity in cylindrical coordinates, and it holds at all points of the fluid free from sources and sinks.

Example 6. A mass of fluid is in motion so that the lines of motion lie on the surface of co-axial cylinders. Show that the equation of continuity is

$$\frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial\theta}(\rho u) + \frac{\partial}{\partial z}(\rho v) = 0,$$

where u, v are the velocity perpendicular and parallel to z.

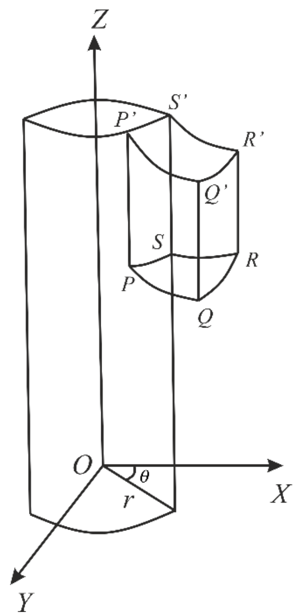
Solution. Consider a fluid particle P, whose cylindrical coordinates are (r, θ, z). With P as one corner construct an element (curvilinear parallelepiped $PQRS, P'Q'R'S'$) with edges

$$PQ = \delta r \; PS = r\delta\theta \text{ and } PP' = \delta z.$$

Let ρ be the density of the fluid at P.

Since the lines of motion lie on the surface of co-axial cylinder, there is no motion along PQ.

Hence the rate of the excess of the flow-in overflow-out along PQ vanishes.



Again, we have

Rate of excess of flow-in over flow-out along $PS = -r\delta\theta \frac{\partial}{r\,\partial\theta}(\rho u \delta r \delta z)$

Rate of excess of flow-in over flow-out along $PP' = -\delta z \frac{\partial}{\partial z}(\rho v r \delta\theta \delta r)$

Again, the rate of increase in mass of the element $= \frac{\partial}{\partial t}(\rho r \delta\theta \delta r \delta z)$

Hence the equation of continuity is given by

$$\frac{\partial}{\partial r}(\rho r \delta\theta \delta r \delta z) = -\delta\theta \frac{\partial}{\partial\theta}(\rho u \delta r \delta z) - \delta z \frac{\partial}{\partial z}(\rho v r \delta\theta \delta r)$$

or

$$r\delta\theta\delta r\delta z \frac{\partial\rho}{\partial t} + \delta r\delta\theta\delta z \frac{\partial}{\partial\theta}(\rho u) + r\delta\delta\delta\theta\delta z \frac{\partial}{\partial z}(\rho v) = 0$$

or

$$\frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial\theta}(\rho u) + \frac{\partial}{\partial z}(\rho v) = 0.$$

7.6 *SUMMARY*

This unit explains the following topics:

(i) Definition of Two-Dimensional Flow.

(ii) Cylindrical Polar Coordinates.

7.7 *GLOSSARY*

(i) Fluid

(ii) Two-Dimensional Flow

(iii) Cylindrical Polar Coordinates

7.8 *REFERENCES AND SUGGESTED READINGS*

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7.9 TERMINAL QUESTIONS

1. Given a velocity field $u = 2x$ and $v = -2y$ show that it satisfies the continuity equation.
2. Consider a velocity field $u = -y$ and $v = x$. Determine if this velocity field satisfies the continuity equation.
3. Given a velocity field $u = x+y$ and $v = 2x-y$, verify if it satisfies the continuity equation.
4. Define two-dimensional flow in fluid dynamics.
5. Explain the use of cylindrical polar coordinates in studying fluid flow.
6. Write the relation between cartesian coordinate system and cylindrical polar coordinate system.

   Solution:
   $$x = r\cos(\theta)$$
   $$y = r\sin(\theta)$$
   $$z = z$$
7. How does the continuity equation in cylindrical coordinates simplify for incompressible, axisymmetric flow?

   Solution: $\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{\partial u_z}{\partial z} = 0.$

UNIT 8: THE STREAM FUNCTION

CONTENTS:

8.1 *INTRODUCTION*

In fluid mechanics, the concepts of stream function play crucial roles in the analysis and understanding of fluid flow patterns. The stream function is a mathematical tool used to visualize and describe the flow field by representing the streamline flow, which is the paths traced by fluid particles as they move through the flow. It provides information about the direction, convergence, divergence, and circulation of the flow.

So, this unit begins by introducing the stream function with their property, leads to the definition of the complex potential.

8.2 *OBJECTIVES*

After completion of this unit learners will be able to:

(i) Define the concept of stream function.

(ii) Describe the physical significance of stream function.

(iii) Describe the elementary theory of functions of a complex variable

(iv) Define the Complex Potential

(v) Derive the Cauchy-Riemann Equations in polar form

8.3 *STREAM FUNCTION AND THEIR PROPERTY*

8.3.1 STREAM FUNCTION

In fluid dynamics, the stream function is a mathematical concept used to describe the flow field of an incompressible fluid. It is a scalar field that represents the streamlines of the fluid flow.

Let u and v be the components of velocity in two-dimensional motion. Then the differential equation of lines of flow or streamline is

$$dx/u = dy/v \text{ or } vdx - udy = 0 \tag{8.1}$$

and the equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \text{ or } \frac{\partial v}{\partial y} = \frac{\partial(-u)}{\partial x} \tag{8.2}$$

(8.2) shows that L.H.S. of (8.1) must be an exact differential, $d\psi$ (say). Thus, we have

$$vdx - udy = d\psi = (d\psi/\partial x)dx + (\partial\psi/\partial y)dy \tag{8.3}$$

so that $u = -\partial\Psi/\partial y$ and $v = \partial\Psi/\partial x$ (8.4)

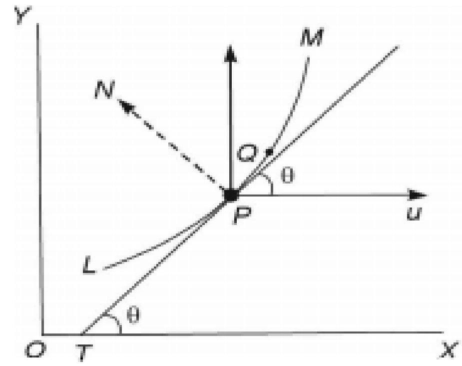
This function ψ is known as the stream function. Then using (8.1) and (8.3), the streamlines are given by $d\psi = 0$ i.e., by the equation $\psi = c$, where c is an arbitrary constant. Thus, the stream function is constant along a streamline. Clearly the current function exists by virtue of the equation of continuity and incompressibility of the fluid.

8.3.2 PHYSICAL SIGNIFICANCE OF STREAM FUNCTION

Let LM be any curve in the $x - y$ plane and let ψ_1 and Ψ_2 be the stream functions at L and M respectively. Let P be an arbitrary point on LM such that arc $LP = s$ and let Q be a neighbouring point on LM such that arc $LQ = s + \delta s$. Let θ be the angle between tangent at P and the x-axis. If u and be the velocity-components at P, then velocity at P along inward drawn normal PN $= \cos\theta - u\sin\theta$ (8.5)

When ψ is the stream function, then we have

$$u = -\partial\psi/\partial y \quad \text{and} \quad v = \partial\psi/\partial x \tag{8.6}$$



Also from Calculus,

$$\cos\theta = dx/ds \text{ and } \sin\theta = dy/ds \tag{8.7}$$

Using (8.5), we get flux across PQ from right to left $= (\cos\theta - u\sin\theta)\delta s$

∴ Total flux across curve LM from right to left

$$= \int_{\text{LM}} (\cos\theta - u\sin\theta) ds = \int_{\text{LM}} \left(\frac{\partial\psi}{\partial x}\frac{dx}{ds} + \frac{\partial\psi}{\partial y}\frac{dy}{ds}\right) ds, \text{ using (8.6) and (8.7)}$$

$$= \int_{\text{LM}} \left(\frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy\right) = \int_{\psi_1}^{\psi_2} d\psi = \psi_2 - \Psi_1$$

One characteristic of the stream function is that the flow across any line that connects two places is represented by the difference of their values at those points.

Important Note:

1. If the stream function exists, it is a possible case of fluid flow satisfies continuity equation which may be rotational flow or irrotational flow.

2. If the stream function satisfies the Laplace equation, it is a case of steady irrotational flow.

3. Stream function represents streamline and it is constant along streamline.

4. The difference between any two stream function give discharge per unit depth.

8.3.3 SPIN COMPONENTS IN TERMS OF STREAM FUNCTION ψ

We know that the velocity components u and are functions of x, y and t and $w = 0$ in twodimensional flow. Hence the spin components $(\xi, \eta, \zeta\ )$ are given by

$$2\xi = \frac{\partial w}{\partial y} - \frac{\partial}{\partial z} = 0 \qquad\qquad 2\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0$$

and $$2\zeta = \frac{\partial}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial y}\right) = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}$$

Let the motion be irrotational so that $\zeta = 0$ also. Then we obtain

$$\partial^2\Psi / \partial x^2 + \partial^2\psi / \partial y^2 = 0 \text{ or } \nabla^2\psi = 0$$

showing that Ψ satisfies Laplace's equation.

8.3.4 SOME ASPECTS OF ELEMENTARY THEORY OF FUNCTIONS OF A COMPLEX VARIABLES

Suppose that $z = x + iy$ and that $w = f(z) = \phi(x, y) + i\psi(x, y)$ where x, y, ϕ, ψ are all real and $i = \sqrt{-1}$. Also, suppose that ϕ and ψ and their first derivatives are everywhere continuous within a given region. If at any point of the region specified by z the derivative $dw/dz (= f'(z))$ is unique, then w is said to be analytic or regular at that point. If the derivative is unique throughout the region, then w is said to be analytic or regular throughout the region. It can be shown that the necessary and sufficient conditions for w to be analytic at z are

$$\partial\phi / \partial x = \partial\psi / \partial y \text{ and } \partial\phi / \partial y = -\partial\psi / \partial x$$

which are known as the Cauchy-Riemann equations. The functions ϕ, ψ are known as conjugate functions.

8.4 *COMPLEX POTENTIAL*

Let $w = \phi + i\psi$ be taken as a function of $x + iy\, i.e,,$ z Thus, suppose that $w = f(z) i.e.$

$$\phi + i\psi = f(x + iy) \quad (8.8)$$

Differentiating (8.8) w.r.t, x and y respectively, we get

$$\partial\phi/\,\partial x + i(\partial\psi/\,\partial x) = f'(x + iy) \quad (8.9)$$

and

$$\partial\phi/\,\partial y + i(\partial\psi\psi/\,\partial y) = if'(x + iy)$$

or

$$\partial\phi/\,\partial y + i(\partial\psi/\,\partial y) = i\{\partial\phi/\,\partial x + i(\partial\psi/\,\partial x)\}, \text{ by } (8.9)$$

Equating real and imaginary parts, we get

$\partial\phi/\,\partial x = \partial\psi/\,\partial y$ and $\partial\phi/\,\partial y = -\hat{\partial}\Psi/\,\partial x$

which is Cauchy-Riemann equations. Then w is an analytic function of z and w is known as the complex porential.

Conversely, if w is an analytic function of z, then its real part is the velocity potential and imaginary part is the stream function of an irrotational two-dimensional motion.

8.4.1 CAUCHY-RIEMANN EQUATIONS IN POLAR FORM

Let $$\phi + i\psi = f(z) = f\left(n'^{\theta}\right) \quad (8.10)$$

Differentiating (8.10) w.r.t. r and θ, we get

$$\frac{\partial\phi}{\partial r} + i\frac{\partial\psi}{\partial r} = f'\left(re^{3\theta}\right)\cdot e^{j\theta} \quad (8.11)$$

and $$\frac{\partial\phi}{\partial\theta} + i\frac{\partial\psi}{\partial\theta} = f'\left(re^{i\theta}\right)\cdot rie^{i\theta} \tag{8.12}$$

From (8.11) and (8.12), we easily obtain

$$\frac{\partial\phi}{\partial\theta} + i\frac{\partial\psi}{\partial\theta} = ir\left(\frac{\partial\phi}{\partial r} + i\frac{\partial\psi}{\partial r}\right)$$

Equating real and imaginary parts, we get

$$\frac{\partial\phi}{\partial\theta} = -r\frac{\partial\psi}{\partial r} \qquad \text{and} \qquad \frac{\partial\psi}{\partial\theta} = r\frac{\partial\phi}{\partial r}$$

Thus, $$\frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \qquad \text{and} \qquad \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\partial\psi}{\partial r} \tag{8.13}$$

which are Cauchy-Riemann equations in polar form.

8.4.2 EXAMPLES BASED ON STREAM FUNCTION

Example 1. If $\phi = A(x^2 - y^2)$ represents a possible flow phenomenon, determine the stream function,

Solution. Here $\phi = A(x^2 - y^2)$ (8.14)

$\therefore$ $\partial\Psi/\partial y = \partial\phi/\partial x = 2Ax$, using (8.14)

Integrating it w.r.t. ' y ', $\psi = 2Axy + f(x)$, (8.15)

where $f(x)$ is an arbitrary function of x. (8.15) gives the required stream function.

Example 2. Determine the stream function $\psi(x, y, t)$ for the given velocity field $u = Ut,\ v = x$,

Solution. We know that $u = -(\partial\psi/\partial y)$ and $v = (\partial\psi/\partial x)$

$$\partial\psi/\partial y = -Ut. \tag{8.16}$$

and $$\partial\psi/\partial x = x \tag{8.17}$$

Integrating (8.16), $\Psi(x, y, t) = -Uty + f(x, t)$. (8.18)

Where $f(x,t)$ is an arbitrary function of x and t.

From (8.18), $\partial\Psi/\partial x = \partial f/\partial x$ (8.19)

Then (8.17) and (8.19) $\frac{\partial f}{\partial x} = x$ (8.20)

Integrating (8.20) $f(x,t) = x^2/2 + F(t)$ (8.21)

where $F(t)$ is an arbitrary function of t.

From (8.18) and (8.21), $\Psi(x, y, t) = -Uty + x^2/2 + F(t)$.

Example 3. The function for a two-dimensional flow is $\phi = x(2y - 1)$. At a point $P(4,5)$ determine the value of stream function.

Solution. Given $\phi = 2xy - x$. (8.22)

Then $u = -\partial\hat{\phi}/\partial x = -2y + 1$ and $v = -\partial\hat{\phi}/\partial \mathrm{y} = -2x$. (8.23)

Now, $u = -\partial\psi/\partial y$ and $v = \partial\psi/\partial x$. (8.24)

From (8.23) and (8.24), $\partial\psi/\partial x = -2x$ and $\partial\psi/\partial y = 2y - 1$.

Now, $d\psi = (\partial\psi/\partial x)dx + (\partial\psi/\partial y)dy = -2xdx + (2y - 1)dy$.

Integrating, $\psi = -x^2 + y^2 - y + C, C$ being constant of integration.

For $\psi = 0$ at the origin, we have Hence $0 = 0 + C$ or $C = 0$.

Hence $\Psi = -x^2 + y^2 - y$

At the point $P(4,5)$, $\psi = -4^2 + 5^2 - 5 = 4$ units.

Example 4. Show that $u = 2cxy,\ = c(a^2 + x^2 - y^2)$ are the velocity components of a possible fluid motion. Determine the stream function.

Solution. Given $u = 2cxy, \quad v = c(a^2 + x^2 - y^2)$ (8.25)

Equation of continuity in xy-plane is given by

$$\partial u/\partial x + \partial v/\partial y = 0 \quad (8.26)$$

From (8.25), $\partial u/\partial x = 2cy$ and $\partial v/\partial y = -2cy$. Putting these values in (8.26) we get $0 = 0$, showing (8.26) is satisfied by u, given by (8.25). Hence u and constitute a possible fluid motion.

Let ψ be the required stream function. Then, we have

$$u = -(\partial\psi/\partial y) \quad \text{or} \quad \partial\psi/\partial y = -2cxy \qquad (8.26)$$

and $$v = \partial\psi/\partial x \quad \text{or} \quad \partial\psi/\partial x = c(a^2 + x^2 - y^2) \qquad (8.27)$$

Integrating (8.26) partially w.r.t. ' y ' $\quad \psi = -cxy^2 + \phi(x,t) \qquad (8.28)$

where $\phi(x,t)$ is an arbitrary function of x and t.

Differentiating (8.28) partially w.r.t. ' x ',

$$\partial\psi/\partial x = -cy^2 + \partial\phi/\partial x \qquad (8.29)$$

(8.27) and (8.29) $\Rightarrow \; -cy^2 + \partial\phi/\partial x = c(a^2 + x^2 - y^2)$

$$\text{or} \qquad \partial\phi/\partial x = c(a^2 + x^2) \qquad (8.30)$$

Integrating (8.30) partially w.r.t. ' x ', $\phi(x,t) = c(a^2x + x^3/3) + \psi(y,t)$, where $\psi(y,t)$ is an arbitrary function of y and t.

Substituting the above value of $\phi(x,t)$ in (8.28), we get

$$\psi = c(ax^2 + x^3/3 - xy^2) + \psi(y,t),$$

which is the required stream function.

Example 5. Show that $u = -\omega y, \; = \omega x, w = 0$ represents a possible motion of inviscid fluid. Find the stream function and sketch stream lines.

Solution. Given $\quad u = -\omega y, \quad v = \omega x \quad$ and $\quad w = 0 \quad (8.31)$

(8.31) $\Rightarrow \partial u/\partial x = 0 = \partial v/\partial y$. Hence the equation of continuity $\partial u/\partial x + \partial v/\partial y = 0$ is satisfied. Hence these exist a two-dimensional motion defined by (8.31).

Now, $$\partial\psi = (\partial\psi/\partial x)dx + (\partial\psi/\partial y)dy \qquad (8.32)$$

But $$\frac{\partial\psi}{\partial x} = -\frac{\partial\phi}{\partial y} == \omega x \quad \text{and} \quad \frac{\partial\Psi}{\partial y} = \frac{\partial\phi}{\partial x} = -u = \omega y \qquad (8.33)$$

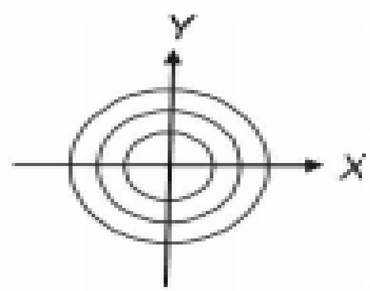
$\therefore$ (8.33) reduces to $d\psi = \omega x dx + \omega y dy = d\{\omega(x^2 + y^2)/2\}$

Integrating, $\Psi = \omega(x^2 + y^2)/2 + c$, where c is an arbitrary constant.

The required streamlines are given by $\Psi = \text{constant} = c'$, say

i.e., $c' = \omega(x^2 + y^2)/2 + c$ or $x^2 + y^2 = 2(c' - c)/w = a^2$, say

Hence the required streamlines are concentric circles with centres at origin as shown in the adjoining figure.



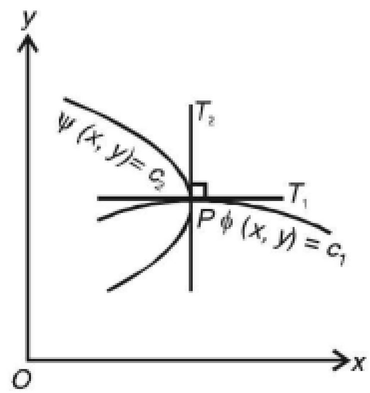
Example 6. To shows that the family of curves $\phi(x, y) = c_1$ and $\psi(x, y) = c_2$; c_1, c_2 being constants, cut orthogonally at their points of intersection.

Solution. Let the curves of constant velocity potential and constant stream function be given by and

$$\phi(x, y) = c_1 \quad (8.34)$$

$$\psi(x, y) = c_2 \quad (8.35)$$

where c_1 and c_2 are arbitrary constants.



Let m_1 and m_2 be gradients of tangents PT_1 and PT_2 at point of intersection P of (8.34) and (8.35). Then, we have

$$m_1 = -\frac{\partial\phi/\partial x}{\partial\phi/\partial y} \text{ and } m_2 = -\frac{\partial\psi/\partial x}{\partial\psi/\partial y} \qquad (8.36)$$

We know that ϕ and ψ satisfy the Cauchy-Riemann equations, namely,

$$\partial\phi/\partial x = \partial\psi/\partial y \text{ and } \partial\phi/\partial y = -\partial\psi/\partial x. \qquad (8.37)$$

Now, from (8.36),

$$m_1 m_2 = \frac{(\partial\phi/\partial x)(\partial\psi/\partial x)}{(\partial\phi/\partial y)(\partial\psi/\partial y)} = \frac{(\partial\psi/\partial y)(\partial\psi/\partial x)}{-(\partial\psi/\partial x)(\partial\psi/\partial y)}, \text{ by (8.37)}$$

Hence $m_1 m_2 = -1$, showing that the curves (1) and (2) cut each other orthogonally.

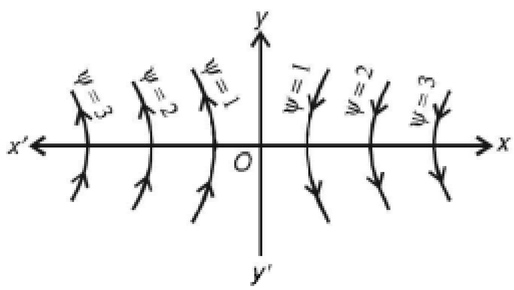
Example 7. The streamlines are represented by (a) $\psi = x^2 - y^2$ and (b) $\psi = x^2 + y^2$ Then (i) determine the velocity and its direction at $(2,2)$, (ii) sketch the streamlines and show the direction of flow in each case.

Solution. Part (a) Given that $\psi = x^2 - y^2$. Now, $u = \partial\psi/\partial y = -2y$ and $v = -\partial\psi/\partial x = -2x$

At (2,2), $u = -4, \text{v} = -4$.

∴ The resultant velocity $= (u^2 + v^{\,2})^{1/2} = (16+16)^{1/2} = 4\sqrt{2}$ units and its direction has a slope $= \mathrm{v}/u = 1$ showing that the velocity vector is inclined at $45°$ to x-axis.

The required streamlines are given by $\psi = c$, where c is a constant, i.e. $x^2 - y^2 = c$, which represents a family of hyperbolas. In figure, we have sketched the steamlines for various values of ψ. The direction of arrowhead shows the direction of flow in each case.



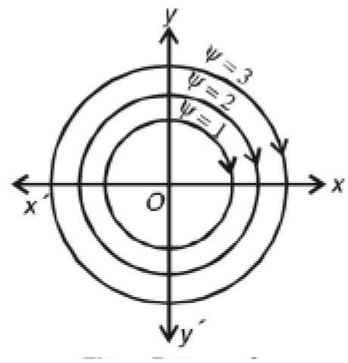
Part (b) Given that $\Psi = x^2 + y^2$

Now, $u = \hat{\partial}\psi/\,\partial y = 2y$, $\mathrm{v} = -\,\partial\psi/\,\partial x = -2x$.

At (2,2), $u = 4$ and $\mathrm{v} = -4$.

∴ The resultant velocity is $(u^2 + \,v^2)^{1/2} = (16+16)^{1/2} =$ $4\sqrt{2}$ units, and its direction has a slope $= \mathrm{v}/u = -1$, showing that the velocity vector is inclined at $135°$ to x-axis.

The required streamlines are given by $\psi = c$, where c is a constant, i.e. $x^2 + y^2 = c$, which represents a family of circles. In figure, we have sketched the streamlines for various values of ψ. The direction of arrowhead shows the direction of flow in each case.



Example 8. If $\phi = 3xy$, find x and y components of velocity at $(1,3)$ and $(3,3)$. Determine the discharge passing between streamlines passing through these points.

Solution. The velocity components u and in x and y directions are given by

$$u = -\partial\phi/\partial x = -3y \text{ and } v = -\partial\phi/\partial y = -3x$$

Hence the velocity components at $(1,3)$ are

$$u = -9, \text{ v} = -3$$

and the velocity components at $(3,3)$ are

$$u = -9, \text{ v} = -9$$

Now, we have and $u = \frac{\partial\psi}{\partial y}, \text{v} = -\partial\psi/\partial x.$

$\Rightarrow$ $\partial\psi/\partial y = -3y$ and $\partial\psi/\partial x = 3x.$

$$d\psi = (\partial\psi/\partial x)dx + (\partial\psi/\partial y)dy = 3xdx - 3ydy$$

Integrating, $y = (3x^2/2) - (3y^2/2) + C$, where C is constant of integration.

Discharge between the streamlines passing through (1,3) and (3,3)

$= \psi(1,3) - \psi(3,3) = (3/2) \times (1-9) - (3/2) \times (9-9) = -12$ units.

Example 9. If the expression for stream function is described by $\psi = x^3 - 3xy^2$, determine whether flow is rotational or irrotational. If the flow is irrotational, then indicate the correct value of the velocity potential.

(a) $\phi = y^3 - 3x^2y$

(b) $\phi = -3x^2y$.

Solution. Now $u = \partial\psi/\partial y = -6xy$, $\text{v} = -\partial\psi/\partial x = -3(x^2 - y^2)$

Hence, $\partial \text{v}/\partial x = -6x$ and $\partial \text{u}/\partial y = -6x$

A two-dimensional flow in xy-plane will be irrotational if the vorticity vector component Ω_z in the z-direction is zero.

Here $\Omega_z = (\partial \text{v}/\partial x) - (\partial u/\partial y) = -6x - (-6x) = 0$

Hence the flow is irrotational.

Now, $u = -\partial\phi/\partial x$ and $\text{v} = -\partial\phi/\partial y$

For an irrotational flow Laplace equation in ϕ must be satisfied,

i.e. $(\partial^2\phi/\partial x^2) + (\partial^2\phi/\partial y^2) = 0$.

We now check the validity of each given value of ϕ.

(a) Given $\phi = y^3 - 3x^2y \Rightarrow \partial^2\phi/\partial x^2 = -6y$ and $\partial^2\phi/\partial y^2 = 6y$

$\therefore (\partial^2\phi/\partial x^2) + (\partial^2\phi/\partial y^2) = -6y + 6y = 0$.

(b) Given $\phi = -3x^2y \;\Rightarrow\; \partial^2\phi/\partial x^2 = -6y$ and $\partial^2\phi/\partial y^2 = 0$

$\therefore (\partial^2\phi/\partial x^2) + (\partial^2\phi/\partial y^2) = -6y + 0 \neq 0.$

Hence the correct value of ϕ is given by $\phi = y^3 - 3x^2y$

Example 10. In a two-dimensional incompressible flow, the fluid velocity components are given by $u = x - 4y$ and $= -y - 4x$. Show that velocity potential exists and determine its form as well as stream function.

Solution. Given $u = x - 4y$ and $\mathrm{v} = -y - 4x$

The velocity potential will exist if flow is irrotational. Therefore, the vorticity component Ω_z in the z-direction must be zero.

Here $\Omega_z = (\partial \mathrm{v}/\partial x) - (\partial u/\partial y) = -4 - (-4) = 0,$

Here the vorticity being zero, the flow is irrotational and so the velocity potential ϕ exists.

Now, we have $d\phi = (\partial\phi/\partial x)dx + (\partial\phi/\partial y)dy = -udx - \mathrm{v}\,dy$

or

$$d\phi = -(x - 4y)dx - (-y - 4x)dy = -xdx + ydy + 4(ydx + xdy)$$

Integration, $\phi = -(x^2/2) + y^2/2 + 4xy + C$, where C is constant of integration.

If $\phi = 0$ at the origin, then, we find $C = 0$.

Hence $\phi = (y^2 - x^2)/2 + 4xy$

Example 11. Find the stream function ψ for a given velocity potential $\phi = cx$, where c is a constant. Also, draw a set of steamlines and equipotential lines.

Solution. The velocity components u and in x and y directions are given by $u = -\partial\phi/\partial x = -c$ and $\mathrm{v} = -\partial\phi/\partial y = 0$.

$\therefore u = -\partial\psi/\partial y$ and $\mathrm{v} = \partial\psi/\partial x$

$\Rightarrow \partial\psi/\partial y = c$ and $\partial\psi/\partial x = 0$.

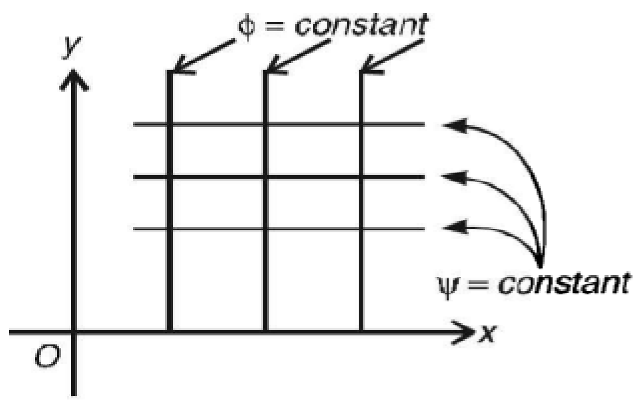
Then, $d\psi = (\partial\psi/\partial x)dx + (\partial\psi/\partial y)dy = cdy$.

Integrating, $\psi = cy + d$,

where d is constant of integration.

Now, $\phi =$ constant $\Rightarrow cx =$ constant $\Rightarrow x =$ constant, showing that the lines of equipotential are parallel to y-axis.

Next, $\psi =$ constant $\Rightarrow cy + d =$ constant $\Rightarrow y =$ constant, showing that the streamlines are parallel to x-axis as shown in the figure.



Example 12. Show that $u = 2cxy,\ = c(a^2 + x^2 - y^2)$ are the velocity components of a possible fluid motion. Determine the stream function.

Solution. Given $u = 2cxy,\ = c(a^2 + x^2 - y^2)$

Equation of continuity in xy-plane is given by $\partial u/\partial x + \partial \mathrm{v}/\partial y = 0$

From, $\partial u/\partial x = 2cy$ and $\partial v/\partial y = -2cy$. Putting these values, we get $0 = 0$, showing equation of continuity is satisfied by u. Hence u and constitute a possible fluid motion.

Let ψ be the required stream function. Then, we have $u = -(\partial\psi/\partial y)$ or $\partial\psi/\partial y = -2cxy$ and $= \partial\psi/\partial x$ or $\partial\psi/\partial x = c(a^2 + x^2 - y^2)$

Integrating partially w.r.t. ' y ', $\psi = -cxy^2 + \phi(x,t)$, where $\phi(x,t)$ is an arbitrary function of x and t.

Differentiating partially w.r.t. ' x ', $\partial\psi/\partial x = -cy^2 + \partial\phi/\partial x$

$\Rightarrow\ -cy^2 + \partial\phi/\partial x = c(a^2 + x^2 - y^2)$ or $\partial\phi/\partial x = c(a^2 + x^2)$

Integrating partially w.r.t. ' x ',

$$\phi(x,t) = c(a^2 x + x^3/3) + \psi(y,t)$$

where $\psi(y,t)$ is an arbitrary function of y and t.

Substituting the above value of $\phi(x,t)$, we get

$\psi = c(\alpha x^2 + x^3/3 - xy^2) + \psi(y,t)$, which is the required stream function.

8.5 *SUMMARY*

This unit explains the following topics:

(i) Definition of Stream Function

(ii) Physical Significance of Stream Function

(iii) Spin Components in Terms of ψ

(iv) Complex Potential

(v) Cauchy-Riemann Equations in polar form

8.6 *GLOSSARY*

(i) Fluid

(ii) Velocity

(iii) Stream Function

(iv) Cauchy-Riemann Equations

(v) Complex Potential

8.7 *REFERENCES AND SUGGESTED READINGS*

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(vi) S.K. Som, G. Biswas, S. Chakraborty (2017), Introduction to Fluid Mechanics and Fluid Machines, McGraw Hill Education; 3rd edition.

8.8 *TERMINAL QUESTIONS*

1. What is a Stream Function?

2. Define the Complex Potential?

3. Explain the Physical Significance of Stream Function?

4. If a stream function exists for the velocity field

$$u = a(x^2 - y^2),\ v = -2axy, \qquad w = 0$$

find it, plot it, and interpret it.

Solution: $\Psi = a\left(x^2 y + \frac{y^3}{3}\right) + c$

5. A velocity field is given by $= x\boldsymbol{i} + (y+t)\boldsymbol{j}$. Find the stream function and the streamlines for this field at $t = 2$.

Solution: $\Psi = xy + 2x + f(2)$

6. Determine the stream function $\psi(x, y, t)$ for the given velocity field $u = -2t^2,\ v = x^3,$

Solution: $t^2 y^2 + \frac{x^4}{4} + F(t)$

7. Determine the stream function corresponding to the velocity potential

a) $\phi = \frac{5}{3}x^3$-$6xy^2$

b) $\phi = x^2$-y^2

c) $\phi = x^3$-$3xy^2$

d) $\phi = -3x^2$+y^3

Solution:

a) $5x^2y - 2y^3 + f(x)$

b) $2xy + f(x)$

c) $3x^2y - y^3 + f(x)$

d) $-2axy + f(x)$

8. **Comment True or False:**

a) The partial derivative of stream function with respect to any direction gives the velocity component perpendicular to that direction.

Solution: True

b) Stream function varies along a streamline.

Solution: False

c) The difference between any two stream function give discharge per unit depth.

Solution: True

d) Stream function is defined as scalar function of space and time.

Solution: True

UNIT 9: STANDARD TWO-DIMENSIONAL FLOWS

CONTENTS:

9.1 *INTRODUCTION*

Two-dimensional flows refer to fluid motions where the velocity components are functions of only two spatial dimensions, x and y. These flows are crucial for simplifying complex three-dimensional problems that are commonly encountered in various engineering applications.

Examples include flow around air-foils, water flow over dams, and airflow in wind tunnels. The simplification to two dimensions allows for analytical and easier computational modelling.

Assuming independence from the third dimension reduces the complexity of the goveing equations, making analytical and numerical solutions more tractable.

9.2 *OBJECTIVES*

After completion of this unit learners will be able to:

(i) Complex Velocity Potential

(ii) Superposition of Flows

9.3 *COMPLEX VELOCITY POTENTIAL*

The complex velocity potential, $W(z)$, is a function that combines the velocity potential, ϕ, and the stream function, ψ, into a single complex function.

$$W(z) = \phi(x, y) + i\psi(x, y)$$

Here, $z = x + iy$ is a complex variable representing the coordinates in the flow field.

The complex velocity potential is related to the complex velocity, $V(z)$, as follows:

$$V(z) = \frac{dW}{dz} = u - iv$$

Where u and v are the velocity componets in the x and y directions, respectively.

Types of standard two-dimensional flows

(i) Uniform flow: A flow where the velocity is constant in both magnitude and direction.

Complex potential: $W(z) = Uz$

Complex velocity: $V(z) = U$

(ii) Source or sink: A point from which fluid emanates or into which fluid converges.

Complex potential: $W(z) = \frac{Q}{2\pi} ln(z)$

Complex velocity: $V(z) = \frac{Q}{2\pi z}$

Here, Q is the source strength (Positive for a source, negative for a sink)

(iii) Vortex: A point around which fluid circulates.

Complex potential: $W(z) = \frac{-i\Gamma}{2\pi} ln(z)$

Complex velocity: $V(z) = \frac{i\Gamma}{2\pi z}$

Here, Γ is the circulation strength.

(iv) Doublet: A combination of source and sink of equal strength but opposite in sign, placed infinitesimally close together.

Complex potential: $W(z) = \frac{\mu}{z}$

Complex velocity: $V(z) = -\frac{\mu}{z^2}$

Here, μ is the doublet strength.

9.4 *SUPERPOSITION OF FLOWS*

The principle of superposition allows us to combine simple flow solutions to create more complex flow patterns. In potential flow theory, since the governing equations (Laplace's equation for incompressible, irrotational flow) are linear, the sum of solutions is also a solution. Here, we explore the superposition of various fundamental flows to model more complicated scenarios.

Note: Uniform Flow and Source/Sink

When a uniform flow is superimposed with a source or sink, the resulting flow pattern is altered by the addition of radial outflow or inflow from the source/sink.

- **Complex Potential:** $W(z) = Uz + \frac{Q}{2\pi} ln(z)$
- **Flow Characteristics:**
  (i) Far from the source, the flow behaves like a uniform flow.
  (ii) Near the source, the flow diverges (source) or converges (sink) radially.
- **Streamline Pattern:** The streamlines show parallel lines far from the origin but curve outward (for a source) or inward (for a sink) as they approach the origin.

Note: Uniform Flow and Doublet (Flow around a Cylinder)

The combination of a uniform flow and a doublet can model the flow around a cylinder. This is a classic problem in fluid dynamics with significant applications in aerodynamics and hydrodynamics.

- **Complex Potential:** $\boldsymbol{W(z) = Uz - \frac{k}{z}}$**,** where k is the strength of the doublet, related to the radius R of the cylinder by $k = UR^2$.
- **Flow Characteristics:**
  (i) At large distances, the flow resembles a uniform flow.
  (ii) Close to the cylinder, the doublet induces a circular flow pattern.

(iii) There are stagnation points on the surface of the cylinder where the flow velocity is zero.

- **Streamline Pattern:** The streamlines wrap around the cylinder, with two stagnation points on the surface of the cylinder at angles $\theta = 0$ and $\theta = \pi$.

Note: Source and Sink Pair (Doublet)

The combination of a source and a sink of equal and opposite strengths, placed very close to each other, forms a doublet. This configuration is useful in modelling flow around bodies with sharp changes in geometry.

- **Complex Potential:** $\boldsymbol{W(z) = \frac{Q}{2\pi}(ln(z-a) - ln(z+a)) \approx -\frac{k}{z}}$ for small a, where Q is the strength of the source/sink, and $2a$ is the distance between them.
- **Flow Characteristics:**

(i) Flow lines originate from the source and terminate at the sink.

(ii) When the source and sink are close together, their effects combine to create a dipole-like pattern.

- **Streamline Pattern:** The streamlines are symmetric about the axis joining the source and sink, showing the characteristic dipole pattern.

Note: Vortex and Uniform Flow

(Flow around a Rotating Cylinder)

Combining a vortex with a uniform flow models the flow around a rotating cylinder, which is important in understanding lift generation in aerodynamics (Magnus effect).

- **Complex Potential:** $W(z) = Uz - \frac{k}{z} - \frac{i\Gamma}{2\pi} ln(z)$,where Γ is the circulation around the cylinder.
- **Flow Characteristics:**

(i) The uniform flow dictates the overall direction.

(ii) The doublet represents the solid boundary of the cylinder.

(iii) The vortex adds rotational motion to the flow, modifying pressure distribution around the cylinder.

- **Streamline Pattern:** The streamlines show the wrapping around the cylinder combined with circulation. The direction of circulation (clockwise or counterclockwise) influences the location of the stagnation points and alters the symmetry of the streamlines.

9.5 *SOLVED EXAMPLES*

Problem1: Consider a uniform flow with velocity $U = 5\text{m/s}$ in the positive x-direction. A doublet of strength $\mu = 10\text{m}^2/\text{s}$ is placed at the origin. Determine the complex potential and velocity at the point $z = 1 + i$.

Solution:

Uniform flow

Complex potential: $W_{uniform}(z) = Uz = 5z$

Velocity: $V_{uniform}(z) = 5$

Doublet

Complex potential: $W_{doublet}(z) = \frac{\mu}{z} = \frac{10}{z}$

Velocity: $V_{doublet}(z) = -\frac{\mu}{z^2} = -\frac{10}{z^2}$

Combined flow

Total complex potential: $W(z) = W_{uniform}(z) + W_{doublet}(z) = 5z + \frac{10}{z}$

Total complex velocity: $V(z) = V_{uniform}(z) + V_{doublet}(z) = 5 + -\frac{10}{z^2}$

Now, evaluation of total complex potential at $z = 1 + i$

$\Rightarrow \bar{z} = 1 - i$

$\Rightarrow z^2 = (1+i)^2 = 2i$

Total complex potential

$W(z) = W_{uniform}(z) + W_{doublet}(z)$

$= 5(1+i) + \frac{10}{(1+i)}$

$= 5(1+i) + \frac{10(1-i)}{(1+i)(1-i)}$

$= 5(1+i) + 5(1-i) = 10$

Now, evaluation of total complex velocity at $z = 1 + i$

$V(z) = V_{uniform}(z) + V_{doublet}(z) = 5 + -\frac{10}{2i} = 5 + 5i.$

So, at $z = 1 + i$, total complex potential is 10 and total complex velocity $5 + 5i$.

Problem2: A source of strength $Q = 8\text{m}^2/\text{s}$ and a vortex with circulation $\Gamma = -4\pi\text{m}^2/\text{s}$ are placed at the origin. Calculate the complex potential and velocity at $z = 2 + 2i$.

Solution:

Complex potential: $W_{source}(z) = \frac{Q}{2\pi} ln(z) = \frac{8}{2\pi} ln(z) = \frac{4}{\pi} ln(z)$

Velocity: $V_{source}(z) = \frac{Q}{2\pi z} = \frac{8}{2\pi z} = \frac{4}{\pi z}$

Vortex:

Complex potential: $W_{vortex}(z) = \frac{-i\,\Gamma}{2\pi} ln(z) = \frac{-i\,(-4\pi)}{2\pi} ln(z) = 2i\; ln(z)$

Velocity: $V_{vortex}(z) = \frac{i\,\Gamma}{2\pi z} = \frac{i\,(-4\pi)}{2\pi z} = \frac{-2i}{z}$

Combined flow:

Total complex potential

$W(z) = W_{source}(z) + W_{vortex}(z) = \frac{4}{\pi} ln(z) + 2i\, ln(z)$

Total complex velocity

$V(z) = V_{source}(z) + V_{vortex}(z) = \frac{4}{\pi z} + \frac{-2i}{z}$

Now, evaluation of total complex potential at $z = 2 + 2i$

$ln(z) = ln(2 + 2i)$

To find above value convert $z = 2 + 2i$ into polar form (Use $x = rcos(\theta), y = rsin(\theta)$)

$\Rightarrow r = \sqrt{2^2 + 2^2} = 2\sqrt{2}$

$and\ \ \theta = \tan^{-1}(\frac{2}{2}) = \frac{\pi}{4}$

So, $ln(2 + 2i) = ln(2\sqrt{2}) + i\frac{\pi}{4} = \ln(2) + \ln(\sqrt{2}) + i\frac{\pi}{4}$

$= \ln(2) + \frac{\ln(2)}{2} + i\frac{\pi}{4} \quad = \frac{3\ln(2)}{2} + i\frac{\pi}{4}$

Now, total complex potential

$W_{source}(z) + W_{vortex}(z) = \frac{4}{\pi} ln(z) + 2i\, ln(z)$

$= \frac{4}{\pi}(\frac{3\ln(2)}{2} + i\frac{\pi}{4}) + 2i\,(\frac{3\ln(2)}{2} + i\frac{\pi}{4})$

$= \frac{6\ln(2)}{\pi} + 3i\, ln(2) + i - \frac{\pi}{2}$

And total complex velocity

$\mathrm{V(z)} = V_{source}(z) + V_{vortex}(z) = \frac{4}{\pi z} + \frac{-2i}{z}$

$$= \frac{4-2i\pi}{\pi(2+2i)}$$

$$= \frac{(4-2i\pi)(2-2i)}{\pi(2+2i)(2-2i)} = \frac{(4-2i\pi)(1-i)}{4\pi}$$

$$= \frac{2(2+\pi-2i-i\pi)}{4\pi} = \frac{(2+\pi)}{2\pi} - \frac{i(2+\pi)}{2\pi} = \frac{(2+\pi)}{2\pi}(1-i)$$

Problem3: Consider a uniform flow with a velocity $\mathrm{U} = 3\mathrm{m/s}$ in the positive x-direction. Determine the complex potential and the velocity at the point $\mathrm{z} = 1 + 2\mathrm{i}$.

Solution:

Complex potential:

$W(z) = Uz = 3z$

At $\mathrm{z} = 1 + 2\mathrm{i}$

$W(z) = Uz = 3(1 + 2\mathrm{i}) = (3 + 6\mathrm{i})$

Complex Velocity:

$$\mathrm{V(z)} = \frac{\mathrm{dW}}{\mathrm{dz}} = \mathrm{U}$$

Here, V(z) =3 (constant for uniform flow)

So, at $\mathrm{z} = 1 + 2\mathrm{i}$, the complex potential is $(3 + 6\mathrm{i})$ and velocity is 3 m/s in the positive x-direction.

Problem 4: A source of the strength $Q = 4m^2/s$ is placed at the origin. Determine the complex potential and the velocity at the point $z = 2i$.

Solution:

Complex potential, for a source, $W(z) = \frac{Q}{2\pi} ln(z)$

$$= \frac{4}{2\pi} ln(z) = \frac{2}{\pi} ln(z)$$

At $z = 2i$:

Convert $z = 2i$ into polar form, then

$z = 2i = 2e^{i\pi/2}$

taking logarithm on both sides, we get

$ln(z) = ln(2i) = ln(2e^{i\pi/2})$

$\Rightarrow \ln(2) + i\frac{\pi}{2}$

So, $W(2i) = \frac{2}{\pi} ln(z) = \frac{2}{\pi}(\ln(2) + i\frac{\pi}{2})$

$$= \frac{2}{\pi}\ln(2) + i$$

Complex Velocity:

$$V(z) = \frac{Q}{2\pi z} = \frac{4}{2\pi z} = \frac{2}{\pi z}$$

At $z = 2i$, $V(2i) = \frac{2}{\pi(2i)} = -\frac{i}{\pi}$.

Problem 5: A vortex with circulation $\Gamma = -2\pi m^2/s$ is placed at the origin. Determine the complex potential and velocity at the point $z = 1$.

Solution:

Complex potential:

$$W_{vortex}(z) = \frac{-i\,\Gamma}{2\pi} ln(z) = \frac{-i\,(-2\pi)}{2\pi} ln(z) = i\, ln(z)$$

At $z = 1$

$$W_{vortex}(z) = i\, ln(1) = 0$$

Velocity:

$$V_{vortex}(z) = \frac{i\,\Gamma}{2\pi z} = \frac{i\,(-2\pi)}{2\pi z} = \frac{-i}{z}$$

At $z = 1$

$$V_{vortex}(z) = \frac{-i}{z} = \frac{-i}{1} = -i\ .$$

Problem 6: A doublet of strength $\mu = 5\text{m}^2/\text{s}$ is placed at the origin. Determine the complex potential and the velocity at the point $z = 1 - i$.

Solution:

Complex potential:

For doublet, $W(z) = \frac{\mu}{z} = \frac{5}{z}$

At $z = 1 - i$, $W(z) = \frac{5(1+i)}{(1-i)(1+i)}$

$$= \frac{5(1+i)}{2}$$

Complex Velocity:

$$V(z) = \frac{-\mu}{z^2} = \frac{-5}{z^2}$$

At $z = 1 - i$, $V(z) = \frac{-5}{(1-i)^2} = \frac{5}{2i} = \frac{5}{2i} \times \frac{-i}{-i} = \frac{5i}{2}$.

Multiple Choice Questions (MCQs)

1. The complex velocity potential for a uniform flow in the x −direction is:

(a) $\frac{Q}{2\pi} ln(z)$

(b) Uz

(c) $\frac{-k}{z}$

(d) $\frac{-i\,\Gamma}{2\pi} ln(z)$

$Answer$: (b)

2. Which of the following represents the complex velocity potential for a vortex?

(a) $\frac{Q}{2\pi} ln(z)$

(b) Uz

(c) $\frac{-k}{z}$

(d) $\frac{-i\,\Gamma}{2\pi} ln(z)$

$Answer$: (d)

3. The velocity components for a source of strength Q are:

(a) $u = \frac{Q}{2\pi r} cos\theta,\ v = \frac{Q}{2\pi r} sin\theta$

(b) $u = \frac{k}{r^2} cos\theta,\ v = -\frac{k}{r^2} sin\theta$

(c) $u = U,\ v = 0$

(d) $u = -\frac{\Gamma}{2\pi r} sin\theta,\ v = \frac{\Gamma}{2\pi r} cos\theta$

$Answer$: (a)

4. The stream function for a doublet is given by

(a) $\psi = Uy$

(b) $\psi = \frac{Q}{2\pi}\theta$

(c) $\psi = -\frac{k}{r} sin\,\theta$

(d) $\psi = -\frac{\Gamma}{2\pi} ln(r)$

$Answer: (c)$

5. In a flow around a cylinder, the complex potential is a combination of which two flows?

(a) Uniform flow and source

(b) Uniform flow and doublet

(c) Source and vortex

(d) Doublet and vortex

$Answer: (b)$

9.6 *SUMMARY*

This unit explains the following topics:

(i) Complex Velocity Potential
(ii) Superposition of Flows

9.7 *GLOSSARY*

(i) Fluid
(ii) Velocity
(iii) Stream Function
(iv) Cauchy-Riemann Equations

9.8 *REFERENCES AND SUGGESTED READINGS*

(i) M. D. Raisinghanai (2013), *Fluid Dynamics*, S. Chand & Company Pvt. Ltd.

(ii) Frank M. White (2011), *Fluid Mechanics*, McGraw Hill.

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(iv) P.K. Kundu, I.M. Cohen & D.R. Dowling (2015), Fluid Mechanics, Academic Press; 6th edition.

(v) F.M. White & H. Xue (2022), Fluid Mechanics, McGraw Hill; Standard Edition.

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UNIT 10: TWO – DIMENSIONAL IMAGE SYSTEM

CONTENTS:

10.1 *INTRODUCTION*

In fluid dynamics, the method of images is a powerful technique used to solve problems involving flow around the boundaries. The method involves introducing imaginary source, sinks, vortices, or other flow elements (termed images) to satisfy boundary conditions. This technique is particularly useful in two-dimensional potential flow problems, where it helps in maintaining the condition imposed by solid boundaries.

Basic concept: When a fluid flow encounters a solid boundary, the normal component of the velocity at the boundary must be zero (impermeability condition). To satisfy this condition, we introduce image elements that cancel out the normal velocity induced by the real flow elements at the boundary.

For example,

1. Image of a source with respect to a line.

Suppose that images of the source m at $A(a, 0)$ on x −axis is required with respect to OY. Take an equal source at $A'(-a, 0)$. Let P be any point on OY such that $AP = A'P = r$. Then the velocity at P due to source at A is m/r along AP and velocity at P due to source A' is m/r along $A'P$. Let PL be perpendicular to OY. Then, we see that.

Resultant velocity at P due to sources at A and A' along PL

$= (m/r)cos\theta -= (m/r)cos\theta$ =0, showing that there will be no flow across OY.

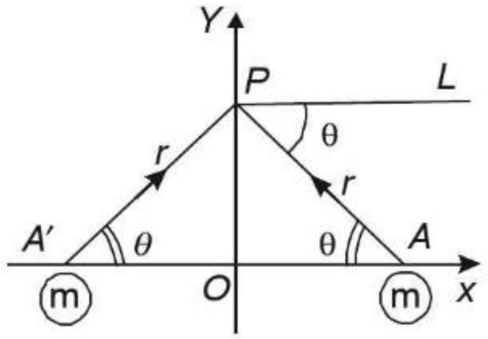


Fig. 10.1.1

Hence, the image of a simple source with respect to a line in two-dimensions is an equal source equidistant from the line opposite to the source.

2. Image of a doublet with respect to a line.

Let PQ be a doublet with its axis inclined at an angle α to OX. Then by using the above result for finding the images of source and sink with respect to OY, we see that the image of the doublet PQ is again an equal doublet $P'Q'$ symmetrically placed as shown in the figure.

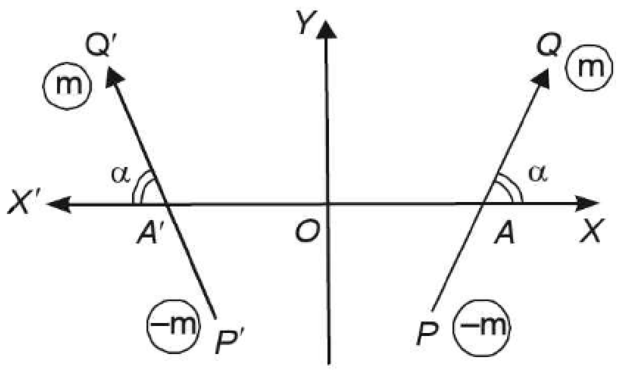


Fig. 10.1.2

10.2 *OBJECTIVES*

After completion of this unit learners will be able to:

(i) The Milne-Thompson circle theorem

10.3 *COMPLEX POTENTIAL FOR IMAGE SYSTEM*

For a point source at $z_0$and its image at $\overline{z_0}$,the complex potential W(z) is:

$$W(z) = \frac{Q}{2\pi}(\ln(z - z_0) - ln(z - \overline{z_0}))$$

10.3.1 *THE MILNE-THOMPSON CIRCLE THEOREM*

The Milne-Thompson circle theorem is a method used to solve flow problems involving circular boundaries. This theorem is particularity useful for flow around cylinder or within circular domains.

Statement: Let f(z) be the complex potential for a flow having no rigid boundaries and such that there are no singularities of flow within the circle $|z| = a$. then, on introducing the solid circular cylinder $|z| = a$ into the flow, the new complex potential is given by $w = f(z) + \bar{f}\left(\frac{a^2}{z}\right)$ for $|z| \geq a$.

Proof: Let C be the cross section of the circular cylinder $|z| = a$. Then on C, $z\bar{z} = a^2$ or $\bar{z} = \frac{a^2}{z}$. hence for points on the circle, we have

$$w = f(z) + \bar{f}\left(\frac{a^2}{z}\right) = f(z) + \bar{f}(\bar{z}) \quad \text{or} \quad \emptyset + i\varphi = f(z) + \bar{f}(\bar{z}) \qquad (1)$$

Since the quantity on R.H.S. of (1) is purely real, equating imaginary parts (1) gives $\varphi = 0$ on C. hence C is a stream line in the new flow.

By hypothesis all the singularities of $f(z)$ (at which sources, sinks, doublets or vortices may be present) lie outside the circle $|z| = a$ and so the singularities of $f\left(\frac{a^2}{z}\right)$ lie inside the circle $|z| = a$. Hence the singularities of $\bar{f}\left(\frac{a^2}{z}\right)$ also lie inside the circle $|z| = a$. Thus, we find that the additional term $\bar{f}\left(\frac{a^2}{z}\right)$ introduces no new singularities into the flow outside the circle $|z| = a$.

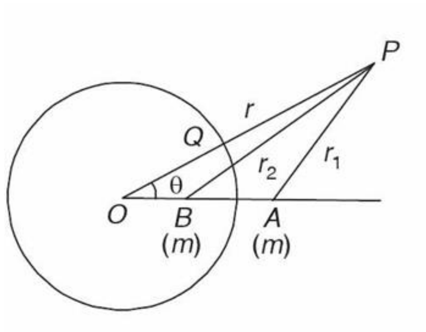
Hence $|z| = a$ is a possible boundary for the new flow $w = f(z) + \bar{f}\left(\frac{a^2}{z}\right)$ is the appropriate complex potential for the new flow.

- **To determine image system for a source outside a circle (or a circular cylinder) of radius a with help of the circle theorem.**

Let $OA = f$. Suppose there is a source of strength m at A where $z = f$, outside the circle of radius a whose centre is at O. When the source is alone in the fluid the complex potential at a point $P(z)$ is given by

$f(z) = -mlog(z - f)$ then $\overline{f(z)} = -mlog(z - f)$

$$\bar{f}\left(\frac{a^2}{z}\right) = -mlog\left(\frac{a^2}{z} - f\right)$$



When the circle of section $|z| = a$ is introduced, then the complex potential in the region $|z| \geq a$ is given by

$w = f(z) + \bar{f}\left(\frac{a^2}{z}\right) = -mlog(z - f) - -mlog\left(\frac{a^2}{z} - f\right)$

$= -mlog(z - f) - mlog\left(\frac{a^2 - fz}{z}\right)$

$= -mlog(z - f) - mlog(a^2 - fz) + mlogz$

$= -mlog(z - f) - mlog(-f)\left(z - \frac{a^2}{f}\right) + mlogz$

$= -mlog(z - f) - mlog\left(z - \frac{a^2}{f}\right) + mlogz - mlog(-f)$

$w = -mlog(z - f) - mlog\left(z - \frac{a^2}{f}\right) + mlogz + \text{constant},$ (1)

The constant (real or complex, $-mlog(-f)$) being immaterial from the view point of analysing the flow. (1) shows that w is the complex potential of

(i) a source m at A, $z = f$

(ii) a source m at B, $z = \frac{a^2}{f}$

(iii) a sink $-m$ at the origin.

Since $OA.OB = a^2$, A and B are the inverse points with respect to the circle $|z| = a$ and so B is inside the circle.

Thus, the image system for a source outside a circle consists of an equal source at inverse point and an equal sink at the centre of the circle.

10.3.2 *APPLICATIONS OF THE THEOREM*

(i) Flow around a cylinder:

Consider a uniform flow $W(z) = Uz$.

The complex potential around a cylinder of the radius R is:

$$W'(z) = Uz + U\left(\frac{R^2}{\bar{z}}\right) = U\left(z + \frac{R^2}{\bar{z}}\right)$$

This represents the flow around a cylinder, where the term $\frac{R^2}{\bar{z}}$ accounts for the effect of the cylinder.

(ii) Source near a cylinder:

Consider a source of strength Q located at $z = a$.

The complex potential due to the source is

$$W(z) = \frac{Q}{2\pi}\ln(z - a)$$

The complex potential around a cylinder is:

$$W(z) = \frac{Q}{2\pi}\ln(z - a) + \frac{Q}{2\pi}ln\left(\frac{R^2}{\bar{z}} - a\right)$$

10.3.3 *Detailed steps for applying the Milne-Thompson circle theorem*

Step 1. Identify the original flow potential $W(Z)$:

Determine the complex potential for the given flow problem without the cylinder.

Step 2. Calculate the image potential:

Compute $W\left(\frac{R^2}{\bar{z}}\right)$, where $\frac{R^2}{\bar{z}}$ represents the image point inside the cylinder.

Step 3. Superpose the potentials:

The resultant complex potential $W'(z)$ is the sum of the original potential and the image potential.

10.4 *SOLVED PROBLEMS*

Problem1: Determine the complex potential for a uniform flow of the speed U past a cylinder of radius R centred at the origin.

Solution: Original potential: $W(z) = Uz$.

Image potential: $W\left(\frac{R^2}{\bar{z}}\right) = U\left(\frac{R^2}{\bar{z}}\right)$

Resultant potential: $W'(z) = Uz + U\left(\frac{R^2}{\bar{z}}\right) = U\left(z + \frac{R^2}{\bar{z}}\right)$

Problem2: Determine the complex potential for a source of the strength Q located at $z = a$ near a cylinder of radius R centred at the origin.

Solution: Original potential: $W(z) = \frac{Q}{2\pi}\ln(z - a)$

Image potential: $W\left(\frac{R^2}{\bar{z}}\right) = \frac{Q}{2\pi} ln\left(\frac{R^2}{z} - a\right)$

Resultant potential: $W'(z) = \frac{Q}{2\pi}\ln(z - a) + \frac{Q}{2\pi} ln\left(\frac{R^2}{z} - a\right)$.

Problem3: Find image of a line source in a circular cylinder.

Solution: Let there be a uniform line source of strength m per unit length through the point $z = c$, where $z > a$. then the complex potential at a point z is given by

$$f(z) = -m\log(z - c)$$

Then $\overline{f(z)} = -m\log(z - c)$

And so $\bar{f}\left(\frac{a^2}{z}\right) = -m\log\left(\frac{a^2}{z} - c\right)$

Let a circular cylinder of section $|z| = a$ be introduced. Then the new complex potential by Milne-Thompson Circle Theorem is given by

$w = f(z) + \bar{f}\left(\frac{a^2}{z}\right)$ for $|z| \geq a$

I.e. $w = -m\log(z - c) - m\log\left(\frac{a^2}{z} - c\right)$

$w = -m\log(z - c) - m\log\left(z - \frac{a^2}{c}\right) + m\log z +$ constant, (1)

The constant (real or complex) being immaterial for the discussion of the flow. The point $z = \frac{a^2}{c}$ is the inverse point of the point $z = c$ with regards to the circle $|z| = a$. Hence (1) shows that the image of a line source in a right circular cylinder is an equal source through the inverse pint in the circular section in the plane of flow together with an equal sink through the centre of the section.

Problem4: Determine image of a line doublet parallel to the axis of a right circular cylinder.

Solution: Let there be a uniform line doublet of strength μ per unit length through the point $z = c > a$. Furthermore let the axis of the line doublet be inclined at an angle α to x −axis. Then the complex potential at a point z is given by

$$f(z) = \left(\mu e^{i\alpha}\right)/(z - c)$$

Then $$\bar{f}(z) = \left(\mu e^{-i\alpha}\right)/(z - c)$$

And so

$$\bar{f}\left(\frac{a^2}{z}\right) = \left(\mu e^{-i\alpha}\right)/\left(\frac{a^2}{z} - c\right)$$

Let a circular cylinder of section $|z| = a$ be introduced. Then he new complex potential by Milne-Thompson Circle Theorem is given by

$w = \left(\mu e^{-i\alpha}\right)/(z - c) + \left(\mu e^{-i\alpha}\right)/\left(\frac{a^2}{z} - c\right).$

Multiple Choice Questions (MCQs)

1. The method of image is used to:

(a) Increase the flow velocity

(b) Satisfy boundary conditions

(c) Change the flow direction

(d) Reduce the flow viscosity

Answer: (b) Satisfy boundary conditions

2. The complex potential for a uniform flow past a cylinder using the Milne-Thompson Circle Theorem is:

(a) Uz

(b) $\frac{Q}{2\pi}\ln z$

(c) $U(z + \frac{R^2}{\bar{z}})$

(d) $-\frac{\kappa}{z}$

Answer: (c) $U\left(z + \frac{R^2}{\bar{z}}\right)$

3. In the Milne-Thompson Circle Theorem, the image point of z for a cylinder of radius R is:

(a) $\frac{R}{z}$

(b) $\frac{R^2}{z}$

(c) Rz

(d) $\frac{z}{R^2}$

Answer: (b) $\frac{R^2}{z}$

4. The complex potential for a source of strength Q located at $z = a$ near a cylinder of radius R is:

(a) $\frac{Q}{2\pi}\ln(z-a)$

(b) $\frac{Q}{2\pi}\left(\ln(z-a) + ln\left(\frac{R^2}{z} - a\right)\right)$

(c) $\frac{Q}{2\pi}\ln(\frac{R^2}{z})$

(d) $\frac{Q}{2\pi}\ln(\frac{R^2}{a})$

Answer: (b) $\frac{Q}{2\pi}\left(\ln(z-a) + ln\left(\frac{R^2}{z} - a\right)\right)$

5. The purpose of introducing image vortices in a flow near a wall is to:

(a) Increase the strength of the original vortex

(b) Satisfy the no- penetration boundary condition.

(c) Create additional vortices in the flow

(d) Change additional vortices in the flow

Answer: (b) Satisfy the no- penetration boundary condition

10.5 *SUMMARY*

This unit explains the following topics:

(i) Complex Velocity Potential

(ii) Superposition of Flows

10.6 *GLOSSARY*

(i) Fluid

(ii) Velocity

(iii) Stream Function

(iv) Cauchy-Riemann Equations

10.7 *REFERENCES AND SUGGESTED READINGS*

(i) M. D. Raisinghanai (2013), *Fluid Dynamics*, S. Chand & Company Pvt. Ltd.

(ii) Frank M. White (2011), *Fluid Mechanics*, McGraw Hill.

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Course Name: FLUID MECHANICS

Course Code: MAT604

BLOCK-IV

SOURCES, SINKS AND DOUBLETS AND STOKES FUNCTION

UNIT 11: INTRODUCTION OF SOURCES

CONTENTS:

11.1 *INTRODUCTION*

A point from which the liquid is emitted radially and symmetrically in all directions is called source. A point to which fluid is flowing symmetrically and radially in all directions is called sink. Sink flow is the reverse of source flow.

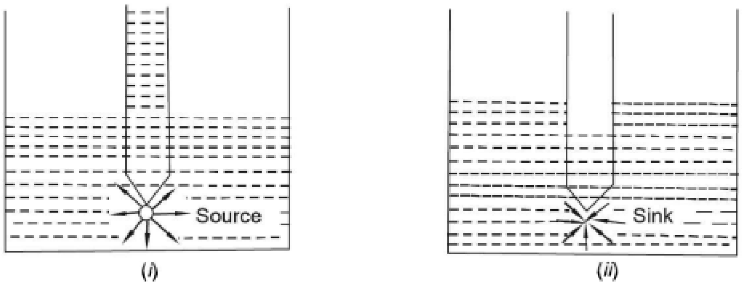


Fig. 11.1

Sources and sinks may arise due to some external causes rather than being easily obtained by certain dynamic effects of the motion of the fluid. For example, consider a simple source in a tank filled with a fluid. This source may be created by taking a long tube of a very small cross-section and injecting fluid through it into the tank as shown in Fig. 11.1 (i). In such a situation, we find that the fluid is coming out from the tube radially into the tank. Again, a sink can be created by taking a long tube of a very small cross-section and sucking fluid through the tube from the tank as shown in Fig. 11.1 (ii).

Consider a source at the origin. Then, the strength of a source is defined as the total volume of flow coming out from the origin in a unit time. Similarly, the amount of fluid going into the sink in a unit time is called the strength of the sink.

In two-dimensions, if $2\pi m$ is the total volume of flow across any small circle surrounding the source, then m is called strength of the source. Sink is a source of strength $-m$.

11.2 OBJECTIVES

After completion of this unit learners will be able to:

(i) Doublet in two-dimensions

(ii) Image of a source with respect to a rigid infinite plane

(iii) Stoke's stream function

11.3 *SOURCES AND SINKS IN TWO - DIMENSIONS*

Consider a circle of radius r with source at its centre. The, radial velocity q_r is given by

$$q_r = -\frac{1}{r}\frac{\partial \psi}{\partial \theta} \tag{11.1}$$

$$\text{or } q_r = -\frac{\partial \phi}{\partial r} \tag{11.2}$$

Comparing Eq. (11.1) and Eq. (11.2), we get

$$\frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \tag{11.3}$$

Then, the flow across the circle is $2\pi r q_r$. Hence, we have

$$2\pi r q_r = 2\pi m \text{ or } r q_r = m \tag{11.4}$$

$$\text{or } r\left(-\frac{1}{r}\frac{\partial \psi}{\partial \theta}\right) = m \text{ (by Eq. (11.1))} \tag{11.5}$$

Integrating and omitting the constant of integration, we get

$$\psi = -m\theta \tag{11.6}$$

Using Eq. (11.3) and Eq. (11.4), we obtain as before

$$\phi = -m\log r \tag{11.7}$$

Eq. (11.6) shows that the streamlines are $\theta =$constant, i.e., straight lines radiating from the source. Again, Eq. (11.7) shows that the curves of equi-velocity potential are $r =$ constant, i.e., concentric circles with centre at the source.

11.4 DOUBLET (OR DIPOLE) IN TWO-DIMENSIONS

A doublet is defined as a combination of source $+m$ and sink $-m$ at a small distance δs apart such that the product $m\delta s$ is finite. (Sink $-m$ means sink of strength $-m$).

Strength of doublet: If $m\delta s=\mu=$ finite where $m\to\infty, \delta s\to 0$, then μ is called the strength of the doublet and line δs is called the axis of the doublet and its direction is taken from sink to source.

Example 1: What arrangement of sources and sinks will give rise to the function $w=\log\left(z-\frac{a^2}{z}\right)$. Prove that two of the streamlines subdivide into a circle $r=a$ and axis of y .

Solution. Given $$w=\log\left(z-\frac{a^2}{z}\right)=\log\left[\frac{z^2-a^2}{z}\right]=\log\left[\frac{(z-a)(z+a)}{z}\right]$$

$$\text{or } w=\log(z-a)+\log(z+a)-\log z$$

which shows that there are two sinks of unit strength at the points $z=a$ and $z=-a$ and a source of unit strength at the origin. Since $w=\phi+i\psi$ and $z=x+iy$, we obtain

$$\phi+i\psi=\log(x+iy-a)+\log(x+iy+a)-\log(x+iy)$$

$$\Rightarrow \phi+i\psi=\log[(x-a)+iy]+\log[(x+a)+iy]-\log(x+iy)$$

Equating imaginary parts on both sides, we have

$$\psi=\tan^{-1}\frac{y}{x-a}+\tan^{-1}\frac{y}{x+a}-\tan^{-1}\frac{y}{x}, \text{ as } \log(\alpha+i\beta)=\frac{1}{2}\log(\alpha^2+\beta^2)+i\tan^{-1}\frac{\beta}{\alpha}$$

$$= \tan^{-1} \frac{\dfrac{y}{x-a} + \dfrac{y}{x+a}}{1 - \dfrac{y}{x-a} \cdot \dfrac{y}{x+a}} - \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{2xy}{x^2 - y^2 - a^2} - \tan^{-1} \frac{y}{x}$$

$$= \tan^{-1} \frac{\dfrac{2xy}{x^2 - y^2 - a^2} - \dfrac{y}{x}}{1 + \dfrac{2xy}{x^2 - y^2 - a^2} \cdot \dfrac{y}{x}} = \tan^{-1} \frac{y\left(x^2 + y^2 + a^2\right)}{x\left(x^2 + y^2 - a^2\right)}$$

The desired streamlines are given by $\psi = \text{constant} = \tan^{-1}(C)$, i.e.

$$\frac{y\left(x^2 + y^2 + a^2\right)}{x\left(x^2 + y^2 - a^2\right)} = C \qquad (1)$$

When $C = 0$, Eq. (1) reduces to $y = 0$. Thus, $x - axis$ is a streamline. Again, when $C \to \infty$, Eq. (1) reduces to $x\left(x^2 + y^2 - a^2\right) = 0$, i.e., $x = 0$ and $x^2 + y^2 = a^2$ or $r = a$, which are streamlines.

Example 2: There is a source of strength m at $(0,0)$ and equal sinks at $(1,0)$ and $(-1,0)$. Discuss two-dimensional motion.

Solution. Proceed just like Ex. 1. Here, we have

$$w = m\log(z-1) + m\log(z+1) - m\log(z-0)$$

$$\phi + i\psi = m\left[\log(x+iy-1) + \log(x+iy+1) - \log(x+iy)\right]$$

Hence, $\psi = m\left[\tan^{-1}\dfrac{y}{x-1} + \tan^{-1}\dfrac{y}{x+1} - \tan^{-1}\dfrac{y}{x}\right]$ or $\dfrac{\psi}{m} = \tan^{-1}\dfrac{y\left(x^2+y^2+1\right)}{x\left(x^2+y^2-1\right)}$

The desired streamlines are given by $\dfrac{\psi}{m} = \text{constant} = \tan^{-1} C$ i.e.

$$\frac{y\left(x^2 + y^2 + 1\right)}{x\left(x^2 + y^2 - 1\right)} = C \qquad (1)$$

Example 3: Find the stream function of the two-dimensional motion due to two equal sources and an equal sink situated midway between them.

Solution. Let there be two sources of strength m at the points $z=a$ and $z=-a$ and a sink at of same strength at $z=0$ (origin). Then complex potential w due to these sources and sink is given by

$$w=-m\log(z-a)-m\log(z+a)+m\log(z-0)$$

$$\Rightarrow \phi+i\psi=m\log(x+iy)-m\log(x+iy-a)-m\log(x+iy+a)$$

$$\Rightarrow \phi+i\psi=m\log(x+iy)-m\log\{(x-a)+iy\}-m\log\{(x+a)+iy\}$$

$$\Rightarrow \phi+i\psi=m\{(1/2)\times\log(x^2+y^2)+i\tan^{-1}(y/x)\}-m\begin{bmatrix}(1/2)\times\log\{(x-a)^2+y^2\}\\ +i\tan^{-1}\{y/(x-a)\}\end{bmatrix}$$

$$-m\left[(1/2)\times\log\{(x+a)^2+y^2\}+i\tan^{-1}\{y/(x+a)\}\right]$$

Equating imaginary parts on both sides, we get

$$\psi=m\tan^{-1}(y/x)-m\left[\tan^{-1}\{y/(x-a)\}+\tan^{-1}\{y/(x+a)\}\right]$$

$$\Rightarrow \frac{\psi}{m}=\tan^{-1}\left(\frac{y}{x}\right)-\tan^{-1}\frac{\{y/(x-a)\}+\{y/(x+a)\}}{1-\{y/(x-a)\}\{y/(x+a)\}}=\tan^{-1}\left(\frac{y}{x}\right)-\tan^{-1}\frac{2xy}{x^2-y^2-a^2}$$

$$\Rightarrow \frac{\psi}{m}=\tan^{-1}\frac{(y/x)-\{2xy/(x^2-y^2-a^2)\}}{1+(y/x)\{2xy/(x^2-y^2-a^2)\}}\Rightarrow \psi=m\tan^{-1}\frac{y(x^2+y^2+a^2)}{x(a^2-x^2-y^2)}$$

Example 4: In a two dimensional liquid motion ϕ and ψ are the velocity and current functions, show that a second fluid motion exists in which ψ is the velocity potential and $-\phi$ the current function; and prove that if the first motion be due to sources and sinks, the second motion can be built up by replacing a source and an equal sink be a line of doublets uniformly distributed along any curve joining them.

Solution. Since ϕ and ψ are the velocity potential and stream function respectively for the two-dimensional motion, we have

$$\partial\phi/\partial x = \partial\psi/\partial y \quad and \quad \partial\phi/\partial y = -(\partial\psi/\partial x) \tag{1}$$

Again if ψ and $-\phi$ be the velocity potential and stream function respectively for another fluid motion in two-dimensions, then the conditions of the type (1) must be satisfied by ψ and $-\phi$ i.e., we must have

$$\frac{\partial\psi}{\partial x} = \frac{\partial(-\phi)}{\partial y} \quad and \quad \frac{\partial\psi}{\partial y} = -\frac{\partial(-\phi)}{\partial x}$$

$$\Rightarrow \partial\phi/\partial y = -(\partial\psi/\partial x) \quad and \quad \partial\phi/\partial x = \partial\psi/\partial y$$

which is true by virtue of (1).

If follows that if $w = \phi + i\psi$ exists, then $w' = \psi - i\phi = -i(\phi + i\psi) = -iw$, also exists.

Second part: Consider a source of strength m at $A(a,0)$ and a sink of strength $-m$ at $B(-a,0)$. Then, the complex potential function w due to them is given by

$$w = -m\log(z-a) + m\log(z+a) = m\log\{(z+a)/(z-a)\} \tag{2}$$

Join A, B by an arbitrary curve. Then the axis of the doublet on this curve is normal to AB. If w'' be the complex potential due this line of doublet then

$$w'' = \int_A^B \frac{me^{i\pi/2}}{z-t}\,dt = me^{i\pi/2}\log\frac{z-a}{z+a} = mi\log\frac{z-a}{z+a} = -iw$$

The required result now follows from the first part.

11.5 *IMAGES*

If in a liquid, a surface S can be drawn across which there is no flow, then any system of sources, sinks and doublets on opposite sides of this surface is known as the image of the system with regard to the surface. Moreover, if the surface S is treated as a rigid boundary and the liquid removed from one side of it, the motion on the other side will remain unchanged.

As there is no flow across the surface, it must be a streamline. Thus, the fluid flows tangentially to the surface and hence the normal velocity of the fluid at any point of the surface is zero.

11.6 *IMAGE OF A SOURCE WITH RESPECT TO A RIGID INFINITE PLANE (STRAIGHT LINE)*

To determine the image of a source $+m$ at $A(a,0)$ with respect to the straight line OY. Place a source $+m$ at $B(-a,0)$. The complex potential at P due to this system is given by

$$\begin{aligned} w &= -m\log(z-a) - m\log(z+a) \\ &= -m\log(z-a)(z+a) \\ &= -m\log\left(r_1 e^{i\theta_1}.r_2 e^{i\theta_2}\right) \\ &= -m\log\left[r_1 r_2 e^{i(\theta_1+\theta_2)}\right] \end{aligned}$$

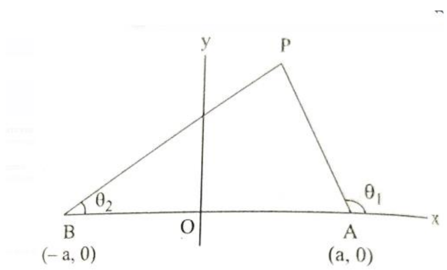


Fig. 11.2

[where $PA = r_1, PB = r_2$]

$$\text{or} \quad \phi + i\psi = -m\left[\log(r_1 r_2) + i(\theta_1 + \theta_2)\right]$$

This implies $\psi = -m(\theta_1 + \theta_2)$ (11.8)

If P lies on $y-axis$, then $PA = PB$ so that $\angle PAB = \angle PBA$,

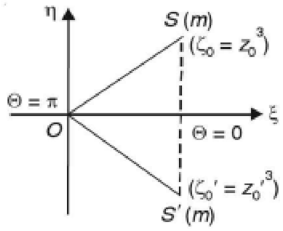
$$\text{i.e., } \pi - \theta_1 = \theta_2 \Rightarrow \theta_1 + \theta_2 = \pi . \qquad (11.9)$$

By Eq. (11.8) and Eq. (11.9),

$$\psi = -m\pi \ or \ \psi = \text{constant}.$$

It means that the $y-axis$ is a stream line. Hence, the image of a source $+m$ at $A(a,0)$ is a source $+m$ at $B(-a,0)$. That is to say, image of a source with respect to a line is a source of the same strength situated on the opposite side of the line at an equal distance.

Example 5: Use the method of images to prove that if there be a source m at the point z_0 in a fluid bounded by the lines $\theta=0$ and $\theta=\pi/3$, the solution is



$$\phi+i\psi=-m\log\left\{\left(z^3-z_0^3\right)\left(z^3-z_0^{'3}\right)\right\} \text{ where } z_0=x_0+iy_0 \text{ and } z_0^{'}=x_0-iy_0.$$

Solution. Consider the following conformal transformation from $z-plane\ (xy-plane)$ to $\zeta-plane\ (\xi\eta-plane)$:

$$\zeta=z^3 \text{ where } z=re^{i\theta} \text{ and } \zeta=Re^{i\Theta} \quad (1)$$

$$\Rightarrow Re^{i\Theta}=r^3e^{3i\theta}\Rightarrow R=r^3 \text{ and } \Theta=3\theta$$

Hence the boundaries $\theta=0$ and $\theta=\pi/3$ in $z-plane$ transform to $\Theta=0$ and $\theta=\pi$ i.e., real axis in $\zeta-plane$. The point z_0 in $z-plane$ transform to point ζ_0 in $z-plane$ such that $\zeta_0=z_0^3$. hence the image system with respect to real axis in $\zeta-plane$ consists of

(i) a source m at $\zeta_0=z_0^3$ (ii) a source m at $\zeta_0^{'}=z_0^{'3}$

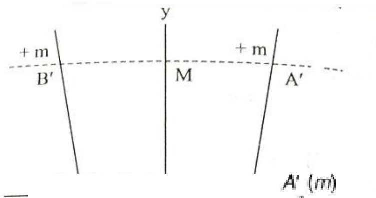
Hence, $w=-m\log\left(\zeta-\zeta_0\right)-m\log\left(\zeta-\zeta_0^{'}\right)$

$$\Rightarrow w=-m\log\left(z^3-z_0^3\right)-m\log\left(z^3-z_0^{'3}\right)$$

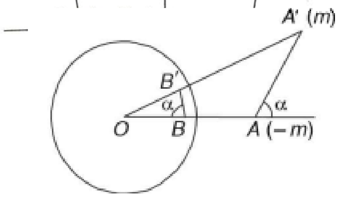
Fig. 11.3

$$\Rightarrow \phi+i\psi=-m\log\left\{\left(z^3-z_0^3\right)\left(z^3-z_0^{'3}\right)\right\}$$

11.7 *IMAGE OF A DOUBLET WITH RESPECT TO A RIGID INFINITE PLANE*



We are to find the image of the doublet AA' with respect to $y-axis$. Treat the doublet AA' as a combination of source $+m$ at A' and sink $-m$ at A with its axis AA' inclined at an angle α with $x-axis$. The images of $-m$ at A and $+m$ at A' with respect to $y-axis$ are respectively $-m$ at B and $+m$ at B' such that $BL = LA, B'M = MA'$. Hence, the image is a doublet BB' of the same strength with its axis anti-parallel to AA'.



11.8 *IMAGE OF A DOUBLET WITH RESPECT TO A CIRCLE*

Let us determine the image of a doublet AA' with its axis making an angle α with OA, outside the circle, there being a sink $-m$ at A and a source m at A' . Join OA and OA'. Let B and B' be the inverse points of A and A' with regard to the circle with O as centre.

Then $$OA \cdot OB = OA' \cdot OB' = a^2, \tag{11.10}$$

where a is the radius of the circle.

Now the image of source m at A' consists of a source m at B' and a sink $-m$ at O. Similarly, the image of sink $-m$ at A consists of a sink at B and a source m at O. Compounding these, we see that source m and sink $-m$ at O cancel each other and hence the image of the given doublet AA' is another doublet BB'.

Let the strength of the given doublet AA' be μ.

Fig. 11.5

Then $$\mu = \lim_{A \to A'} (m \cdot AA') \tag{11.11}$$

From (11.10) $$OA/OA' = OB'/OB, \tag{11.12}$$

showing that triangles OAA' and $OB'B$ are similar. From these similar triangles, we have

$$\frac{BB'}{AA'} = \frac{OB'}{OA} = \frac{OB'}{OA} \cdot \frac{OA'}{OA'} = \frac{a^2}{OA \cdot OA'} \tag{11.13}$$

$$\therefore \mu' = strength\ of\ \ doublet\ B'B = \lim_{B'\to B}\left(m\cdot B'B\right) = \lim_{A\to A'}\frac{a^2}{OA\cdot OA'}\left(m\cdot AA'\right),\ \text{by (11.13)}$$

$\Rightarrow \mu' = \mu a^2 / f^2$, using (11.11) and taking $OA = OA' = f$

Thus the image of a two-dimensional doublet at A with respect to a circle is another doublet at the inverse point B, the axes of the doublets making supplementary angles with the radius OBA.

Example 6: Find image of a line source in a circular cylinder.

Solution. Let there be a uniform line source of strength m per unit length through the point $z = c$, where $z > a$. then the complex potential at a point z is given by

$$f(z) = -m\log(z-c)$$

Then $$\overline{f}(z) = -m\log(z-c)$$

and so $$\overline{f}\left(a^2/z\right) = -m\log\left\{\left(a^2/z\right)-c\right\}$$

Let a circular cylinder of section $|z| = a$ be introduced. Then the new complex potential by Milne-Thomson's circle theorem is given by

$$w = f(z) + \overline{f}\left(a^2/z\right) \quad for \quad |z| \geq a$$

$$\Rightarrow w = -m\log(z-c) - m\log\left\{\left(a^2/z\right)-c\right\}$$

$$\Rightarrow w = -m\log(z-c) - m\log\left\{z-\left(a^2/z\right)\right\} + m\log z + const., \quad (1)$$

The constant (real or complex) being immaterial for the discussion of the flow. The point $z = a^2/c$ is the inverse point of the point $z = c$ with respect to the circle $|z| = a$. Hence, (1) shows that the image of a line source in a right circular cylinder is an equal line source through the inverse point in the circular section in the plane of flow together with an equal line sink through the centre of the section.

11.9 *AXIS-SYMMETRIC FLOWS*

A flow pattern is said to be axisymmetric when it is identical in every plane that passes through a certain straight line. The straight line in question is referred to as the symmetry axis.

11.10 *STOKE'S STREAM FUNCTION*

The equation of continuity in cylindrical coordinates (r,θ,z) is

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho r u)+\frac{1}{r}\frac{\partial}{\partial \theta}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0$$

For liquid, this becomes

$$\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{\partial w}{\partial z}=0$$

If the motion be symmetrical about $z-axis$, then $v=0$ and the equation of continuity becomes

$$\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{\partial w}{\partial z}=0$$

Instead of $z-axis$, if we take $x-axis$ as the axis of symmetry, $\overline{\omega}$ the direction perpendicular to $x-axis$ and u,v velocity components in these directions, then

$$\frac{1}{\overline{\omega}}\frac{\partial}{\partial \overline{\omega}}\left(\overline{\omega}v\right)+\frac{\partial u}{\partial x}=0\left[r\to\overline{\omega},z\to x,w\to u,u\to v\right]$$

$$\text{or}\quad \frac{\partial}{\partial x}\left(u\overline{\omega}\right)+\frac{\partial}{\partial \overline{\omega}}\left(\overline{\omega}v\right)=0\left(i.e.,\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=0\right)$$

But this is the condition that

$$\overline{\omega}v\,dx-u\overline{\omega}\,d\overline{\omega}=\text{ an exact differential }=d\psi\text{, say}$$

$$=\frac{\partial \psi}{\partial x}dx+\frac{\partial \psi}{\partial \overline{\omega}}d\overline{\omega}$$

This implies $\frac{\partial\psi}{\partial x}=\overline{\omega}v,\ \frac{\partial\psi}{\partial\overline{\omega}}=-u\overline{\omega}$

$$\Rightarrow u=-\frac{1}{\overline{\omega}}\frac{\partial\psi}{\partial\overline{\omega}},\ v=\frac{1}{\overline{\omega}}\frac{\partial\psi}{\partial x},$$

This function ψ is called the Stoke's function. The streamlines are given by

$$\frac{dx}{u}=\frac{d\overline{\omega}}{v}\ or\ v\,dx-u\,d\overline{\omega}=0$$

or $\frac{1}{\overline{\omega}}\frac{\partial\psi}{\partial x}dx+\frac{1}{\overline{\omega}}\frac{\partial\psi}{\partial\overline{\omega}}d\overline{\omega}=0$,

or $d\psi=0\ or\ \psi=$ constant along a streamline.

That is why, ϕ is called the Stoke's stream function. Notice that ψ exists even if ϕ does not exist.

11.11 *A PROPERTY OF STOKE'S FUNCTION*

2π times the difference of the values of Stoke's stream function at two points in the same meridian plane is equal to the flow across the angular surface obtained by the revolution around the axis of curve joining the points.

Proof. Let 2π be an element of the curve and θ its inclination to the axis, then outward flow across the surface of the revolution.

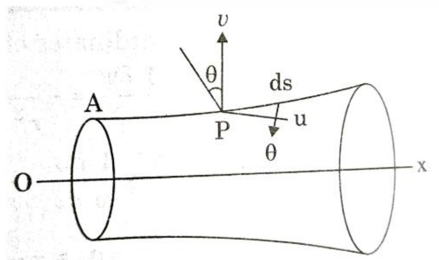


Fig. 11.6

$$=\int_A^B(v\cos\theta-u\sin\theta)2\pi\overline{\omega}\,ds$$
$$=2\pi\int_A^B\left(\frac{\partial\psi}{\partial x}dx+\frac{\partial\psi}{\partial\overline{\omega}}d\overline{\omega}\right)=2\pi\int_A^B d\psi$$
$$=2\pi(\psi_B-\psi_A).$$

This proves the required result.

Example 7: Discuss the motion for which Stoke's stream function is given by

$\psi = \frac{1}{2}V\left[a^4 r^{-2}\cos\theta - r^2\right]\sin^2\theta,$ where r is the distance from a fixed point and θ is the angle, this distance makes with a fixed direction.

Solution. Given $\psi = \frac{1}{2}\frac{Va^4}{r^2}\cos\theta\sin^2\theta - \frac{1}{2}Vr^2\sin^2\theta$ (1)

Evidently, ψ is a sum of two terms. Here, liquid flows with velocity V parallel to $x-axis$ in presence of a fixed solid of revolution.

$$\psi = \frac{1}{2}\frac{Va^4}{r^2}\cos\theta\sin^2\theta$$

is the stream function for a solid which is moving with velocity V parallel to negative direction of $x-axis$. In this case, boundary condition is

$$\psi = \frac{1}{2}Vr^2\sin^2\theta + const.$$

On boundary

$$\frac{1}{2}Vr^2\sin^2\theta + const. = \frac{1}{2}\frac{Va^4}{r^2}\cos\theta\sin^2\theta$$

This implies that $$const. = 0,\ r^2\sin^2\theta = \frac{a^4}{r^2}\cos\theta\sin^2$$

$$\Rightarrow r^4 = a^4\cos\theta.$$

It follows that the given stream function gives the motion of a liquid flowing past a solid $r^4 = a^4\cos\theta$, moving with velocity V along $x-axis$.

Example 8: A and B are a simple source and sink of strengths m and m' respectively in an infinite liquid. Show that the equation of the streamlines is $m\cos\theta - m'\cos\theta' = const.$, where θ, θ' are the angles which AP, BP make with AB, P, being any point. Prove also that if $m > m'$, the cone defined by the equation $\cos\theta = 1 - (2m'/m)$ divides the streamlines issuing from A into two sets, one extending to infinity and the other terminating at B.

Solution. Let Stokes's stream function at any point P be ψ. Then, we have

$$\psi = m\cos\theta - m'\cos\theta'$$

Fig. 11.7

$\therefore$ The required streamlines are given by $\psi = const.$,

i.e., $$m\cos\theta - m'\cos\theta' = const. = c \tag{1}$$

For the extreme streamline leaving A (say at angle α) and leaving B, we find that when P is very near to A, $\theta = \alpha, \theta' = \pi$ and when P is very near B, $\theta = 0$ and $\theta' = 0$. Hence for such streamline, (1) gives

$$m\cos\alpha + m' = m - m' \qquad or \qquad \cos\alpha = 1 - (2m'/m).$$

This generates the cone $\cos\theta = 1 - (2m'/m)$.

11.12 *SUMMARY*

This unit explains the following topics:

(i) Sources and sinks in two-dimensions

(ii) Doublet in two-dimensions

(iii) Image of a source with respect to a rigid infinite plane (straight line)

(iv) Image of a doublet with respect to a rigid infinite plane

(v) Image of a doublet with respect to a circle

(vi) Stoke's stream function

11.13 *REFERENCES AND SUGGESTED READINGS*

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(iii) John Cimbala and Yunus A Çengel (2019), *Fluid Mechanics: Fundamentals and Applications*, McGraw Hill.

(iv) P.K. Kundu, I.M. Cohen & D.R. Dowling (2015), Fluid Mechanics, Academic Press; 6th edition.

(v) F.M. White & H. Xue (2022), Fluid Mechanics, McGraw Hill; Standard Edition.

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11.14 *TERMINAL QUESTIONS*

1. The image of a source $+m$ with respect to a circle is a source $+m$ at the inverse point and

   (i) a source $+m$ at the centre

   (ii) a source $+m$ at the same point

   (iii) a sink $-m$ at the centre

   (iv) None of these.

2. If ψ be the stream function, then the equation of streamline is given by

   (i) $\psi =$ constant

   (ii) $\psi =$ polynomial

   (iii) $\psi =$ trigonometric function

   (iv) $\psi =$ logarithmic function

3. If the fluid is directed radially inward to a point from all directions in a symmetric manner, it is

   (i) Source

   (ii) Doublet

   (iii) Sink

   (iv) Triplet

4. How many sinks are there if the complex potential is given by $w = \log\left\{z - \left(a^2/z\right)\right\}$?

   (i) 1

(ii) 2

(iii) 3

(iv) None of these.

5. With usual notations complex potential of a doublet is

(i) $\mu e^{i\alpha}/(z-a)$

(ii) $\mu e^{-i\alpha}/(z-a)$

(iii) $\mu e^{i\alpha}/(z+a)$

(iv) None of these.

6. In usual notations, the Stokes' stream function for a simple source on the axis of x is

(i) $m\sin\theta$

(ii) mx

(iii) mx/r

(iv) mx/r^2

7. For a simple source of strength m at the origin, the values of Stokes' stream function at the point $P(r,\theta,\phi)$ is

(i) $m\sin\theta$

(ii) $m\cos\theta$

(iii) $m\sin 2\theta$

(iv) $m\cos 2\theta$

8. The relation between ϕ and ψ is

(i) $\partial\phi/\partial y = -(\partial\psi/\partial x)$ *and* $\partial\phi/\partial x = \partial\psi/\partial y$

(ii) $\partial\phi/\partial y = -(\partial\psi/\partial y)$ *and* $\partial\phi/\partial x = \partial\psi/\partial y$

(iii) $\partial\phi/\partial y = -(\partial\psi/\partial x)$ *and* $\partial\phi/\partial y = \partial\psi/\partial y$

(iv) None of these.

9. The family of curves given by $\phi = const.$ and $\psi = const.$ intersect at

(i) 30°

(ii) 45°

(iii) 60°

(iv) 90°

10. The stream function is constant along a particular streamline flow

(i) False statement

(ii) True statement

(iii) Both of above

(iv) None of these

11. Determine the image of a line doublet parallel to the axis of a right circular cylinder.

12. Two sources, each of strength $+m$ are placed at the points $(-a,0),(a,0)$ and a sink of strength $2m$ at the origin. Show that the streamlines are the curve $\left(x^2+y^2\right)^2 = a^2\left(x^2-y^2+\lambda xy\right)$, where λ is a variable parameter.

13. Show that the image system of a source outside a circle consists of an equal source at the inverse point and an equal sink at the centre of the circle.

14. An infinite mass of liquid is moving irrotationally and steadily under the influence of a source of strength μ and an equal sink at a distance $2a$ from it. Prove that the kinetic energy of the liquid which passes in unit time across the plane which bisects at right angles the line joining the source and sink is $\left(8\pi\rho\mu^3\right)/7a^4, \rho$ being the density of the liquid.

15. Show that the force per unit length exerted on a circular cylinder, radius a, due to a source of strength m, at a distance c from the axis is $\left(2\pi\rho m^2 a^2\right)/c\left(c^2-a^2\right)^2$.

16. Show that the image with regard to a sphere of a doublet whose axis passes through the centre is a doublet at the inverse point.

17. Show that a uniform stream of velocity U can be obtained as the limit $a \to \infty$ of the field due to a source of strength $2\pi a^2 U$ at $(-a,0,0)$ and a sink of strength $-2\pi a^2 U$ at $(a,0,0)$.

11.15 *ANSWERS*

1. (iii)
2. (i)
3. (iii)
4. (ii)
5. (i)
6. (iii)
7. (ii)
8. (i)
9. (iv)
10. (i)
11. $w = \dfrac{\mu e^{-i\alpha}}{z-c} + \dfrac{\mu e^{-i\alpha}}{\left(a^2/z\right)-c}$

UNIT 12: RELATION BETWEEN CARTESIAN COMPONENTS

CONTENTS:

12.1 *INTRODUCTION*

Before this unit we have studied the introduction of sources, sinks and doublets, images in rigid infinite plane, axis symmetric flows stoke stream function. In this unit we will read what is the difference between rectangular components of stress. And we also discussed about transnational motion of fluid element. The components of a vector along orthogonal axes are called rectangular Cartesian components. In a simple way when we break down a vector into its simplest parts along straight line (axes) that are at right angles to each other, we call those parts "rectangular components or cartesian components. Think of it like this imagine a vector as an arrow. We can split that arrow into two or three simpler arrows that point along straight lines (like the x and y axes on a graph). Those simpler arrows are the rectangular components. There are two ways to write these components one is using number only (scalar notation) and other is using letters and arrows (Cartesian vector notation).

12.2 *OBJECTIVES*

After studying this unit the learner will be able to

(i) Find the relation between rectangular (Cartesian) components of stress.
(ii) Explain transnational motion of fluid element.
(iii) Describe the stress components in a real fluid.

12.3 *RELATION BETWEEN RECTANGULAR (CARTESIAN) COMPONENTS OF STRESS*

Let us consider the motion of a small rectangular parallelepiped of viscous fluid, its centre being p (x, y ,z) and its edges of lengths $\rho\delta x$, δy, δz, parallel to fixed Cartesian axes, as shown in the figure.

Let ρ be the density of the fluid. The mass $\rho\delta x$, δy, δz of the fluid element remain constant and the element is presumed to move along with the fluid. In the figure, the pont P_1 and P_2 have been taken on the centre of the faces so that they have co ordinates (x- $\frac{\delta x}{2}$, y, z) and (x+ $\frac{\delta x}{2}$, y ,z) respectively.

At P (x, y, z) the force components parallel to OX, OY, OZ on the surface area δy, δz through P and having i as unit are (σ_{xx}, δ_y, δ_z, σ_{xy}, δ_y, δ_z, σ_{xz} δ_y, δ_z).

At P_2 (x + $\frac{\delta x}{2}$, y, z), since i is the unit normal measured outwards from the fluid, the corresponding force components across the parallel plane of area δy, δz, are

$$\left[\sigma_{xx} + \frac{\delta x}{2}\left(d\,\frac{\sigma_{xx}}{dx}\right)\delta_y\delta_{z,}\sigma_{xy} + \frac{\delta x}{2}\left(d\,\frac{\sigma_{xy}}{dx}\right)\,\delta_y\delta_{z,}\sigma_{xy} + \frac{\delta x}{2}\left(d\,\frac{\sigma_{xx}}{dx}\right)\delta_y\delta_{z,}\right]$$

For the parallel plane through P_1 (x- $\frac{\delta x}{2}$,y, z) since –i is the unit normal drawn outward from the fluid element, the corresponding component are

$$\left[-\sigma_{xx} - \frac{\delta x}{2}\left(\partial\,\frac{\sigma_{xx}}{\partial x}\right)\delta_y\delta_{z,} - \sigma_{xy} - \frac{\delta x}{2}\left(\partial\,\frac{\sigma_{xy}}{\partial x}\right)\,\delta_y\delta_{z,} - \sigma_{xz} - \frac{\delta x}{2}\left(\partial\,\frac{\sigma_{xx}}{\partial x}\right)\delta_y\delta_{z,}\right]$$

The forces on the parallel plane through P_1 and P_2 are equivalent to a single force at P with components

$$\left[\frac{\partial\sigma_{xx}}{\partial x},\frac{\partial\sigma_{xy}}{\partial x},\frac{\partial\sigma_{xz}}{\partial x}\right]\,\delta_{x,}\delta_y\delta_{z,}$$

Together with couples whose moments (upto third order terms) are

$$\begin{cases}-\sigma_{xx\ \delta_x\ \sigma_y\sigma_z}\ \ about\ Oy,\\ \ \ \sigma_{xy}\delta_x\delta_y\delta_z about\ Oz\end{cases}$$

Similarly the pair of faces perpendicular to the y axis give a force at P having components

$$\left[\frac{\partial\sigma_{yx}}{\partial x},\frac{\partial\sigma_{yy}}{\partial y},\frac{\partial\sigma_{yz}}{\partial y}\right]\,\delta_{x,}\delta_y\delta_{z,}$$

Together with couple of moments

$$\begin{cases}-\sigma_{yx\ \delta_x\ \sigma_y\sigma_z}\ \ about\ Oy,\\ \ \ \sigma_{yz}\delta_x\delta_y\delta_z about\ Oz\end{cases}$$

The pair of faces perpendicular to the z- axis give a force at P having components

$$\left[\frac{\partial\sigma_{zx}}{\partial z},\frac{\partial\sigma_{zy}}{\partial z},\frac{\partial\sigma_{zz}}{\partial z}\right]\,\delta_{x,}\delta_y\delta_{z,}$$

Together with couple of moments

$$\begin{cases} -\sigma_{zy}\,\delta_x\,\sigma_y\sigma_z \ \ about\ Oy, \\ \ \sigma_{zx}\delta_x\delta_y\delta_z about\ Oz \end{cases}$$

Combining the surface forces of all six faces of the parallelepiped, we observe that they reduce to a single force at P having the components

$$\left[\left(\frac{\partial\sigma_{xx}}{\partial x},\frac{\partial\sigma_{yz}}{\partial y},\frac{\partial\sigma_{zx}}{\partial z}+\right),\left(\frac{\partial\sigma_{xy}}{\partial x},\frac{\partial\sigma_{yy}}{\partial y},\frac{\partial\sigma_{zy}}{\partial z}+\right),\left(\frac{\partial\sigma_{xz}}{\partial x},\frac{\partial\sigma_{yz}}{\partial y},\frac{\partial\sigma_{zz}}{\partial z}+\right)\right]\delta_{x,}\delta_y\delta_{z,}$$

Together with a vector couple having Cartesian components

$$\left[\left(\sigma_{yz}-\sigma_{zy}\right),\left(\sigma_{zx}-\sigma_{xz}\right),\left(\sigma_{xy}-\sigma_{yx}\right)\right]\delta_{x,}\delta_y\delta_{z,}$$

Now suppose the external body forces acting at P are [X, Y, Z] per unit mass, so that the total body force on the element has components [X, Y, X] $\rho\delta_{x,}\delta_y\delta_z$. Let us take moments about i-direction through P. then, we have

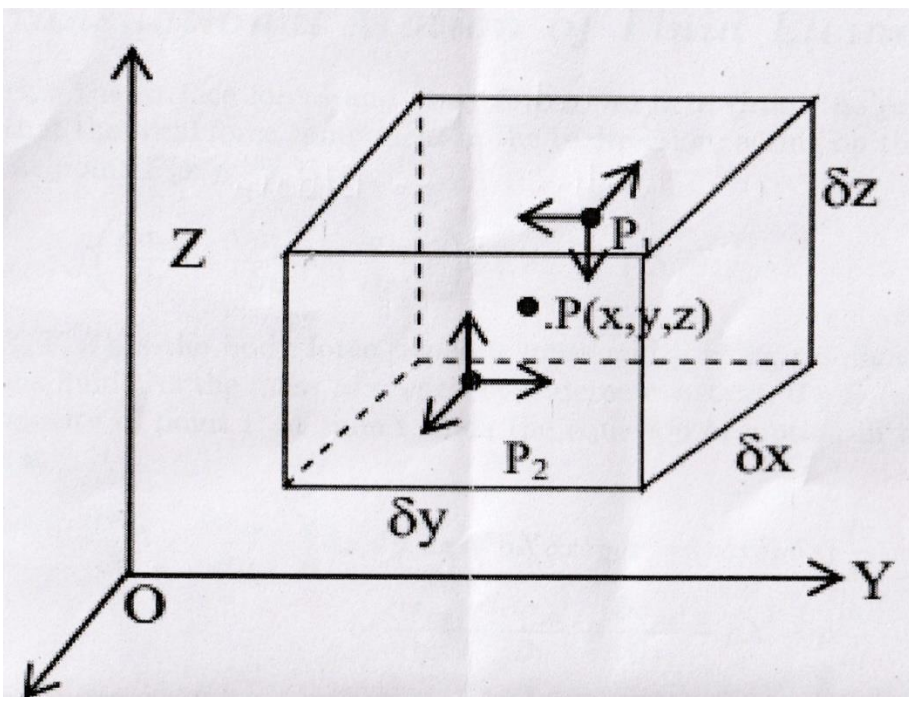

Fig. 12.3.1

Total moment of forces = moments of inertia about axis × angular acceleration

i.e. $(\sigma_{yz}-\sigma_{zy})\,\delta_{x,}\delta_y\delta_{z,}$ + terms of 4th order in $\delta_{x,}\delta_y\,\delta_z$ = terms of 5th order in $\delta_{x,}\delta_y\delta_{z.}$

thus, to the third of smallness in $\delta_{x,}\delta_y\delta_z$ we obtain

$(\sigma_{yz} - \sigma_{zy})\,\delta_{x,}\delta_y\delta_z = 0$

Hence the considered fluid elements becomes vanishingly small, we obtain

$(\sigma_{yz} = \sigma_{zy}$

Similarly, we get

$$\sigma_{zx} = \sigma_{xz}, \qquad \sigma_{xy} = \sigma_{yz}$$

Thus, the stress matrix is diagonally symmetric and contains only six unknowns. In other words, we have proved that

$\Sigma_{ij} = \sigma_{ji}$, (I, j = x,y,z)

i.e. σ_{ij} is symmetric.

In fact, σ_{ij} is a symmetric second order Cartesian tensor.

12.4 *TRANSNATIONAL MOTION OF FLUID ELEMENT*

Considering the surface forces and body forces, we note (from the previous article) that the total force component in the i-direction, acting on the fluid elements at point P (x,y,z), is

$$\left(\frac{\partial\sigma_{xx}}{\partial x}, \frac{\partial\sigma_{yz}}{\partial y}, \frac{\partial\sigma_{zx}}{\partial z} +\right)\delta_{x,}\delta_y, \delta_z + X\rho\delta_{x,}\delta_y\delta_{z,} \qquad (1)$$

Where X, Y, Z is the body force per unit mass and ρ being the density of the viscous fluid. As the mass $\rho\delta_{x,}\delta_y\delta_z$ is considered constant, if q = (u, v, w) be the velocity of point P at time t, then the equation of motion in the i- direction is

$$\left(\frac{\partial\sigma_{xx}}{\partial x}, \frac{\partial\sigma_{yz}}{\partial y}, \frac{\partial\sigma_{zx}}{\partial z} +\right)\delta_{x,}\delta_y, \delta_z + \rho X \delta_{x,}\delta_y\delta_{z,} = (\rho X \delta_{x,}\delta_y\delta_z)\frac{du}{dt} \qquad (2)$$

Or $\left(\frac{\partial\sigma_{xx}}{\partial x}, \frac{\partial\sigma_{yz}}{\partial y}, \frac{\partial\sigma_{zx}}{\partial z} +\right) + \rho X = \rho\frac{du}{dt}$ (3)

If u = u (x, y, z, t), then

$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$ where $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + q.\delta$ thus (3) becomes

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X + \frac{1}{\rho}\left(\frac{\partial \sigma_{xx}}{\partial x}, \frac{\partial \sigma_{yz}}{\partial y}, \frac{\partial \sigma_{zx}}{\partial z}\right) \quad (4)$$

Similarly, the equations of motion in k j and directions are

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = Y + \frac{1}{\rho}\left(\frac{\partial \sigma_{xx}}{\partial x}, \frac{\partial \sigma_{yy}}{\partial y}, \frac{\partial \sigma_{zx}}{\partial z}\right) \quad (5)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = Z + \frac{1}{\rho}\left(\frac{\partial \sigma_{xx}}{\partial x}, \frac{\partial \sigma_{yz}}{\partial y}, \frac{\partial \sigma_{zz}}{\partial z}\right) \quad (6)$$

Equation (4) (5) (6) provide the equations of motion of the fluid element at P (x,y,z).

In tensor form, if the coordinates are x_i, the velocity component u_i, the body force components X_i, where I = 1, 2, 3 the equation of motion can be expressed as

$$\frac{\partial u_i}{\partial t} + u_j u_{i,j} = X_i + \frac{1}{\rho}\sigma_{ji,} (i, j = 1,2,3).$$

12.4.1 STRESS COMPONENTS IN A REAL FLUID

Let δs be a small rigid plane area inserted at a point P in a viscous fluid Cartesian co-ordinates (x, y, z) are referred to a set of fixed axes OX, OY, OZ Suppose that δF_n is the force exerted by the moving fluid on one side of δs, the unit vector n being taken to specify the normal at P to δs on this side. We know that in the case of an inviscit fluid, δF_n is aligned with n. For a viscous fluid, however, frictional forces are called into play between the fluid and the surface so that δF_n will also have a component tangential to δs. We suppose the Cartesian components of δF_n to be (δF_{nx} , δF_{ny} , δF_{nz}) so that

$$\delta F_n = \delta F_{nx} i + \delta F_{ny} j + \delta F_{nz} k.$$

Then the components of stress parallel to the axes are defined to be σ_{nx} , σ_{ny} ,σ_{nz} , where

$$\sigma_{nx} = \lim_{\partial s -> 0} \frac{\partial F_{nx}}{\partial s} = \frac{dF_{nx}}{ds}$$

$$\sigma_{ny} = \lim_{\partial s -> 0} \frac{\partial F_{ny}}{\partial s} = \frac{dF_{ny}}{ds}$$

$$\sigma_{nz} = \lim_{\partial s -> 0} \frac{\partial F_{nz}}{\partial s} = \frac{dF_{nz}}{ds}$$

In the components σ_{nx}, σ_{ny}, σ_{nz}, the first suffix n denotes the direction of the normal to the elemental plane δs whereas the second suffix x or y or z denotes the direction in which the component is measured

If we identify n in turn with the unit vectors and in i , j , and k in OX, OY, OZ which is achieved by suitably re-orientating , we obtain the following three sets of stress components.

$\sigma_{xx}, \sigma_{xy}, \sigma_{xz}$;

$\sigma_{yx}, \sigma_{yy}, \sigma_{yz}$;

$\sigma_{zx}, \sigma_{zy}, \sigma_{zz}$.

The diagonal elements σ_{xx}, σ_{yy}, σ_{zz} of this array are called normal or direct stresses. The remaining six elements are called shearing stresses. For an in viscid fluid, we have

$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p$

$\sigma_{xy} = \sigma_{xz} = \sigma_{yx} = \sigma_{yz} = \sigma_{zx} = \sigma_{zy} = 0$

Here, we consider the normal stresses as positive when they are tensile and negative when they are compressive, so that p is the hydrostatic pressure. The matrix

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \qquad (1)$$

is called the stress matrix. If its components are known, we can calculate the total forces on any area at any chosen point. The quantities σ_{ij} (i, j = x, y, z) are called the components of the stress tensor, whose matrix is of the form (1). Further, we observe that σ_{ij} is a tensor of order two.

To show that only six components suffice to determine the state of stress at a point

As indicated in 1.4, the state of stress at a point is fully determined by nine components of the stress tensor: However, as highlighted in 1.1, the stress tensor is symmetric, meaning the off-diagonal components are equal. Therefore, only six components suffice to determine the state of stress at point.

12.5 *SUMMARY*

Translational motion: When an object moves in a straight line and every part of the object moves the same distance, in the same direction and at the same speed.

In viscid fluid: An viscid fluid is a fluid that has no thickness or resistance to flow. In other word it is a fluid that flows easily and smoothly, without any friction.

12.6 GLOSSARY

(i) Fluid

(ii) Two-Dimensional Flow

(iii) Cylindrical Polar Coordinates

(iv) Viscous fluid: is a real fluid that flows with some resistance in the opposite direction of its flow.

(v) Operators

12.7 *REFERENCES AND SUGGESTED READINGS*

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(iii) John Cimbala and Yunus A Çengel (2019), *Fluid Mechanics: Fundamentals and Applications*, McGraw Hill.

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12.8 *TERMINAL QUESTIONS*

1. How can a vector be determined when rectangular component are known?

2. What is the difference between transnational and rotational motion.

3. Define the Relation Between Rectangular Components of Stress.

4. Define Stress Components in A Real Fluid.

5. Explain transnational motion of fluid element.

UNIT 13: *THE RATE OF STRAIN QUADRIC AND PRINCIPAL STRESSES AND ITS PROPERTY*

CONTENTS:

13.1 *INTRODUCTION*

The strain rate at some point within the material measures the rate at which the distances of adjacent parcels of the material change with time in the neighborhood of that point. It comprises both the rate at which the material is expanding or shrinking (expansion rate), and also the rate at which it is being deformed by progressive shearing without changing its volume (shear rate). It is zero if these distances do not change, as happens when all particles in some region are moving with the same velocity (same speed and direction) and/or rotating with the same angular velocity, as if that part of the medium were a rigid body.

13.2 *OBJECTIVES*

After completion of this unit learners will be able to:

(i) Principal Stresses
(ii) Stress Analysis in fluid motion
(iii) The Navier-Stokes Equations of Motion of a viscous Fluid.

13.3 THE RATE OF STRAIN QUADRIC AND PRINCIPAL STRESSES

13.3.1 The rate of Strain Quadric:

The definition of strain rate was first introduced in 1867 by American metallurgist Jade LeCocq, who defined it as "the rate at which strain occurs. It is the time rate of change of strain." In physics the strain rate is generally defined as the derivative of the strain with respect to time. Its precise definition depends on how strain is measured.

The strain is the ratio of two lengths, so it is a dimensionless quantity (a number that does not depend on the choice of measurement units). Thus, strain rate has dimension of inverse time and units of inverse second, s^{-1} (or its multiples).

In mechanics and materials science, strain rate is the time derivative of strain of a material. Strain rate has dimension of inverse time and SI units of inverse second, s^{-1} (or its multiples). The strain rate at some point within the material measures the rate at which the distances of adjacent parcels of the material change with time in the neighborhood of that point. It comprises both the rate at which the material is expanding or shrinking (expansion rate), and also the rate at which it is being deformed by progressive shearing without changing its volume (shear rate). It is zero if these distances do not change, as happens when all particles in some region are moving with the same velocity (same speed and direction) and/or rotating with the same angular velocity, as if that part of the medium were a rigid body.

The strain rate is a concept of materials science and continuum mechanics that plays an essential role in the physics of fluids and deformable solids. In an isotropic Newtonian fluid, in particular, the viscous stress is a linear function of the rate of strain, defined by two coefficients, one relating to the expansion rate (the bulk viscosity coefficient) and one relating to the shear rate (the "ordinary" viscosity coefficient). In solids, higher strain rates can often cause normally ductile materials to fail in a brittle manner.

13.3.2 Principal Stresses

Principal stress is the normal stress acting onto the principal plane that has zero shear stress.

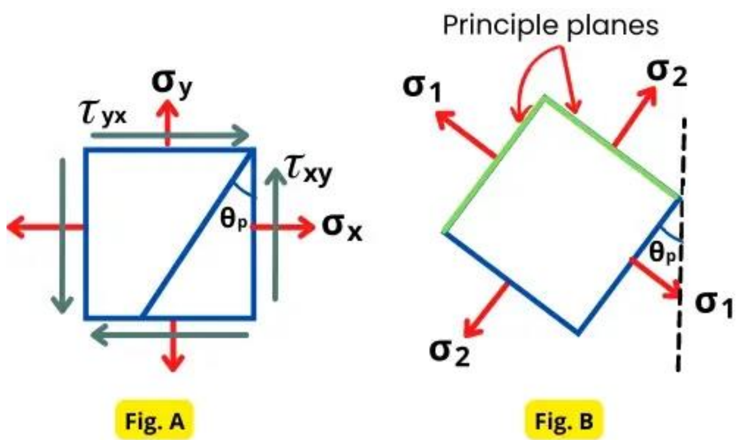

Figure 1

The above figure. A show the member subjected to the axial stresses as well as shear stresses and figure-B shows the principal stresses and principal planes.

A principal plane is an oblique plane in an object that bears no shear stress. The principal planes lie at a principal angle (θp) from the reference plane as shown in figure-B and there is no shear stress acts on it.

The normal stresses acting on that plane (σ1 and σ2) are the principal stresses.

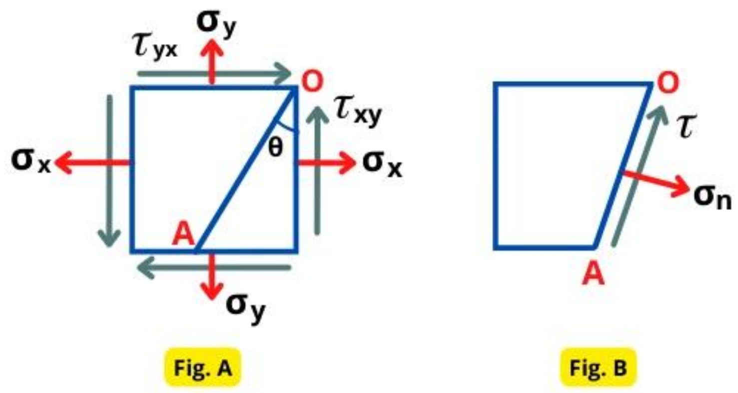

Figure 2

Principal stress explanation:

The member shown in the figure. A is subjected to the axial stresses in the x and y direction $\left(\sigma_x, \sigma_y\right)$ and complementary shear stresses on horizontal and vertical planes $\left(\tau_{xy}, \tau_{yx}\right)$.

Other than horizontal and vertical planes, the member has numerous inclined planes that are inclined at an angle θ (0° to 90°) from the reference plane. These planes are known as oblique planes.

13.4 SOME FURTHER PROPERTIES OF THE RATE OF STRAIN QUADRIC

Strain is a unitless measure of how much an object gets bigger or smaller from an applied load. Normal strain occurs when the elongation of an object is in response to a normal stress (i.e. perpendicular to a surface), and is denoted by the Greek letter epsilon. A positive value corresponds to a tensile strain, while negative is compressive.

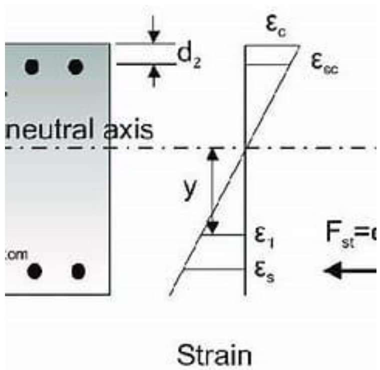

Figure 3

The rate of strain quadric is a concept from fluid dynamics and continuum mechanics that helps in visualizing and analyzing the deformation of fluid elements. It is particularly useful in understanding the local flow properties and the nature of strain in a fluid. Here are the key properties and aspects of the rate of strain quadric:

13.4.1 Definition

The rate of strain quadric is defined by the symmetric part of the velocity gradient tensor, also known as the rate of deformation tensor D, which is given by: D= $\frac{1}{2}(\nabla v + (\nabla v)^T)$ where v is the velocity field of the fluid.

13.4.2 Tensor Representation

The rate of strain tensor D can be written in matrix form as:

$$\begin{pmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{yx} & D_{yy} & D_{yz} \\ D_{zx} & D_{zy} & D_{zz} \end{pmatrix}$$

Since D is symmetric, $D_{ij} = D_{ji}$

13.4.3 Strain Rate Components

The components of the rate of strain tensor represent the rate of change of deformation in various directions:

- $D_{xx}, D_{yy} \, and \, D_{zz}$ are the normal strain rates in the x, y and z directions, respectively.
- $D_{xy} = D_{yx}, D_{xz} = D_{zx} \, and \, D_{yz} = D_{zy}$ are the shear strain rates, which describe the rate of change of the angle between the axes.

13.4.4 Strain Rate Quadric

The rate of strain quadric is a geometric representation of the strain rate tensor. It is given by the quadratic form: $Q(x) = x^T Dx$ where x is a position vector. The quadric provides a visualization of the strain rates in different directions.

13.4.5 Eigenvalues and Eigenvectors

The principal strain rates and principal directions can be determined by finding the eigenvalues and eigenvectors of D:

- The eigenvalues $\lambda_1, \lambda_2, \, and \, \lambda_3$ of D represent the principal strain rates.
- The corresponding eigenvectors give the directions of the principal strains.

13.4.6 Physical Interpretation

- **Normal Strain Rates**: The principal strain rates (eigenvalues) describe the rate at which material elements expand or contract along the principal directions.

- **Shear Strain Rates**: Off-diagonal elements represent the rate at which the shape of the material element is distorted without a change in volume.

The rate of strain quadric can be visualized as an ellipsoid, with the lengths of the principal axes of the ellipsoid proportional to the principal strain rates. The orientation of the axes represents the principal directions of strain.

Understanding the rate of strain quadric is essential in fields such as:

- Fluid mechanics, for analyzing flow patterns and turbulence.
- Continuum mechanics, for studying material deformation.
- Structural engineering, for assessing stress and strain in materials.

The rate of strain quadric provides a comprehensive framework for analyzing the deformation characteristics of fluid elements. It encapsulates both normal and shear strain rates, offering insights into the local behavior of the flow and the nature of material deformation.

13.5 *STRESS ANALYSIS INFLUID MOTION:*

Stress–strain analysis (or stress analysis) is an engineering discipline that uses many methods to determine the stresses and strains in materials and structures subjected to forces. In continuum mechanics, stress is a physical quantity that expresses the internal forces that neighboring particles of a continuous material exert on each other, while strain is the measure of the deformation of the material.

In simple terms, we can define stress as the force of resistance per unit area, offered by a body against deformation. Stress is the ratio of force over area (S = R/A, where S is the stress, R is the internal resisting force and A is the cross-sectional area).

Strain is the ratio of change in length to the original length, when a given body is subjected to some external force (Strain = change in length ÷ the original length).

Stress analysis is a primary task for civil, mechanical and aerospace engineers involved in the design of structures of all sizes, such as tunnels, bridges and dams, aircraft and rocket bodies, mechanical parts, and even plastic cutlery and staples. Stress analysis is also used in the maintenance of such structures, and to investigate the causes of structural failures.

Typically, the starting point for stress analysis are a geometrical description of the structure, the properties of the materials used for its parts, how the parts are joined, and the maximum or typical forces that are expected to be applied to the structure. The output data is typically a quantitative description of how the applied forces spread throughout the structure, resulting in stresses, strains and the deflections of the entire structure and each component of that structure. The analysis may consider forces that vary with time, such as engine vibrations or the load of moving vehicles. In that case, the stresses and deformations will also be functions of time and space.

In engineering, stress analysis is often a tool rather than a goal in itself; the ultimate goal being the design of structures and artifacts that can withstand a specified load, using the minimum amount of material or that satisfies some other optimality criterion.

Stress analysis may be performed through classical mathematical techniques, analytic mathematical modelling or computational simulation, experimental testing, or a combination of methods.

The term stress analysis is used throughout here for the sake of brevity, but it should be understood that the strains, and deflections of structures are of equal importance and in fact, an analysis of a structure may begin with the calculation of deflections or strains and end with calculation of the stresses.

General principles

Stress analysis is specifically concerned with solid objects. The study of stresses in liquids and gases is the subject of fluid mechanics.

Stress analysis adopts the macroscopic view of materials characteristic of continuum mechanics, namely that all properties of materials are homogeneous at small enough scales. Thus, even the smallest particle considered in stress analysis still contains an enormous number of atoms, and its properties are averages of the properties of those atoms.

In stress analysis one normally disregards the physical causes of forces or the precise nature of the materials. Instead, one assumes that the stresses are related to strain of the material by known constitutive equations.

By Newton's laws of motion, any external forces that act on a system must be balanced by internal reaction forces, or cause the particles in the affected part to accelerate. In a solid object, all particles must move substantially in concert in order to maintain the object's overall shape. It follows that any force applied to one part of a solid object must give rise to internal reaction forces that propagate from particle to particle throughout an extended part of the system. With very rare exceptions (such as ferromagnetic materials or planet-scale bodies), internal forces are due to very short range intermolecular interactions, and are therefore manifested as surface contact forces between adjacent particles — that is, as stress.

Fundamental problem

The fundamental problem in stress analysis is to determine the distribution of internal stresses throughout the system, given the external forces that are acting on it. In principle, that means determining, implicitly or explicitly, the Cauchy stress tensor at every point.

The external forces may be body forces (such as gravity or magnetic attraction), that act throughout the volume of a material; or concentrated loads (such as friction between an axle and a bearing, or the weight of a train wheel on a rail), that are imagined to act over a two-dimensional area, or along a line, or at single point. The same net external force will have a different effect on the local stress depending on whether it is concentrated or spread out.

13.6 *RELATION BETWEEN STRESS AND RATE OF STRAIN*

The relationship between stress and rate of strain is a fundamental concept in the study of materials and their deformation behavior, particularly in the field of rheology and solid mechanics.

Stress and Strain

Stress (σ): It is the internal force per unit area within a material. It is typically measured in Pascals (Pa).

Strain (ϵ): It is the deformation or displacement of material per unit length. It is a dimensionless quantity.

Rate of Strain ($\dot{\epsilon}$)

The rate of strain, also known as the strain rate, is the rate at which strain occurs. It is the derivative of strain with respect to time: $\dot{\epsilon} = d\epsilon / dt$. The relationship between stress and rate of strain varies depending on the type of material and its behavior under stress.

The relationship between stress and rate of strain varies depending on the type of material and its behavior under stress.

Elastic Materials

For perfectly elastic materials (e.g., Hookean solids), the relationship is typically between stress and strain, not the rate of strain. Hooke's Law describes this relationship: $\sigma = E \cdot \epsilon$
where E is the Young's modulus of the material.

Viscous Fluids

For purely viscous fluids (e.g., Newtonian fluids), the stress is proportional to the rate of strain: $\sigma = \eta \cdot \dot{\epsilon}$ where η is the viscosity of the fluid.

Viscoelastic Materials

Viscoelastic materials exhibit both elastic and viscous behavior. The relationship between stress and strain rate for such materials can be more complex and is often described using models that combine elements of both elasticity and viscosity. Two common models are:

Maxwell Model: It combines a purely viscous damper and a purely elastic spring in series.

Kelvin-Voigt Model: It combines a purely viscous damper and a purely elastic spring in parallel.

13.7 *THE COEFFICIENT OF VISCOSITY AND LAMINAR FLOW*

When you pour yourself a glass of juice, the liquid flows freely and quickly. But when you pour syrup on your pancakes, that liquid flows slowly and sticks to the pitcher. The difference is fluid friction, both within the fluid itself and between the fluid and its surroundings. We call this property of fluids *viscosity*. Juice has low viscosity, whereas syrup has high viscosity. In the previous sections we have considered ideal fluids with little or no viscosity.

The precise definition of viscosity is based on *laminar*, or nonturbulent, flow. Before we can define viscosity, then, we need to define laminar flow and turbulent flow. Figure shows both types of flow. Laminar flow is characterized by the smooth flow of the fluid in layers that do not mix. Turbulent flow, or turbulence, is characterized by eddies and swirls that mix layers of fluid together.

Figure 4

Figure Smoke rises smoothly for a while and then begins to form swirls and eddies. The smooth flow is called laminar flow, whereas the swirls and eddies typify turbulent flow. If you watch the smoke (being careful not to breathe on it), you will notice that it rises more rapidly when flowing smoothly than after it becomes turbulent, implying that turbulence poses more resistance to flow. (Credit: Creativity103)

Figure shows schematically how laminar and turbulent flow differ. Layers flow without mixing when flow is laminar. When there is turbulence, the layers mix, and there are significant velocities in directions other than the overall direction of flow. The lines that are shown in many illustrations are the paths followed by small volumes of fluids. These are called *streamlines*. Streamlines are smooth and continuous when flow is laminar, but break up and mix when flow is turbulent. Turbulence has two main causes. First, any obstruction or sharp corner, such as in a faucet, creates turbulence by imparting velocities perpendicular to the flow. Second, high speeds cause turbulence. The drag both between adjacent layers of fluid and between the fluid and its surroundings forms swirls and eddies, if the speed is great enough.

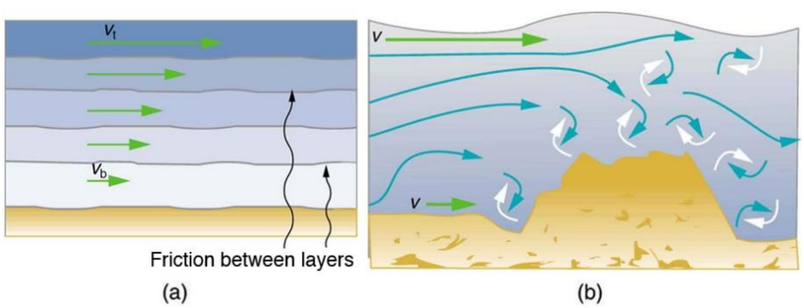

Figure 5

Figure (a) Laminar flow occurs in layers without mixing. Notice that viscosity causes drag between layers as well as with the fixed surface. (b) An obstruction in the vessel produces turbulence. Turbulent flow mixes the fluid. There is more interaction, greater heating, and more resistance than in laminar flow.

13.8 *THE NAVIER-STOKES EQUATIONS OF MOTION OF A VISCOUS FLUID*

Fluid mechanics is the field of physics that deals with the physical mechanics of fluids (plasmas, gases, and liquids) and forces acting on them. It has a wide variety of applications in fields like engineering, oceanography, astrophysics, geophysics, biology, and meteorology. Fluid mechanics can be categorized into fluid statics and fluid mechanics. Fluid statics is the study of fluids at the state of rest. Fluid dynamics is the study of the impacts of forces on fluids in motion. It is a section of continuum mechanics, a field that deals with the matter without concerning the information that comes out of the inherent properties of atoms. It only models matter from a macroscopic perspective rather than from an atomic or molecular viewpoint.

Fluid dynamics is a prolific research field which is generally mathematically complex. Numerous problems are wholly or partly unsolved and are efficiently addressed by numerical techniques, usually using computers. A cutting-edge discipline known as computational fluid dynamics is dedicated to this approach. Particle image velocimetry is an experimental technique for analyzing and visualizing the flow of fluids. It also takes into account the visual nature of the fluid flow.

In fluid mechanics, the Navier-Stokes equations are partial differential equations that express the flow of viscous fluids. These equations are generalizations of the equations developed by Leonhard Euler (18th century) to explain the flow of frictionless and incompressible fluids. In 1821, Claude-Louis Navier put forward the component of viscosity (friction) for a more realistic and difficult problem of viscous fluids. During the entire middle period of the 19th century, George Gabriel Stokes refined this work even though entire solutions were found only in the case of basic two-dimensional flows. The complicated turbulence or vortices, or chaos that happens in three-dimensional fluid flows as velocities rise, has become intractable to any but numerical analysis techniques. The Navier–Stokes equations numerically describe the conservation of mass and the conservation of momentum for Newtonian fluids.

13.9 *SUMMARY*

This unit explains the following topics:

(i) Stress Analysis in fluid motion

(ii) Relation between stress and rate of strain

(iii) The coefficient of viscosity and Laminar flow

13.10 *REFERENCES AND SUGGESTED READINGS*

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13.11 TERMINAL QUESTIONS

Question 1. Which is the most important property used in the rolling process?

(a) Toughness

(b) Hardness

(c) Resilience

(d) Ductility

Answer: (d)

Question 2. Which of the following material does not have a linear portion in the stress-strain curve?

(a) Steel

(b) Magnesium

(c) Grey cast iron

(d) Aluminum

Answer: (c)

Question 3. Calculate the maximum shear strain at the point where principal strains are 100×10^{-6} and -200×10^{-6}.

Answers: 200×10^{-6}.

Question 4. Define Relation between stress and rate of strain.

Question 5. Define coefficient of viscosity and Laminar flow.

Question 6. Define Stress Analysis in fluid motion.

UNIT 14: STOKES FUNCTION

CONTENTS:

14.1 *INTRODUCTION*

In fluid dynamics, the Stokes stream function is used to describe the streamlines and flow velocity in a three-dimensional incompressible flow with axisymmetry. A surface with a constant value of the Stokes stream function encloses a streamtube, everywhere tangential to the flow velocity vectors. Further, the volume flux within this streamtube is constant, and all the streamlines of the flow are located on this surface.

14.2 *OBJECTIVES*

After studying this unit the learner will be able to

(i) Stoke's Stream Function Stokes function.
(ii) Image of source relative to sphere
(iii) Image of doublet relative to sphere

14.3 *STOKE'S STREAM FUNCTION*

The Stokes stream function is a mathematical representation of the trajectories of particles in a steady flow of fluid over an object. It is used to describe the streamlines and flow velocity in a three-dimensional incompressible flow with axisymmetry. A surface with a constant value of the Stokes stream function encloses a streamtube, everywhere tangential to the flow velocity vectors. The stream function depends on the position of the arbitrary point, and, possibly, on that of the fixed point.

Stokes flow also named creeping flow or creeping motion is a type of fluid flow where advective inertial forces are small compared with viscous forces. The Reynolds number is low, i.e. Re <<1. This is a typical situation in flows where the fluid velocities are very slow, the viscosities are very large, or the length-scales of the flow are very small.

Creeping flow was first studied to understand lubrication. In nature, this type of flow occurs in the swimming of microorganisms and sperm. In technology, it occurs in paint, MEMS devices, and in the flow of viscous polymers generally.

If the streamlines in all the planes passing through a given axis are the same, the fluid motion is said to be axi-symmetric. We have already considered such flow for irrotational motion in spherical polar co-ordinates. (r, θ, φ) in which the line , $\theta = 0$ is the axis of symmetry.

Suppose the z-axis be taken as axis of symmetry, then $q_\theta = 0$ and the fluid motion is the same in every plane $\theta = constant$ (meridian plane) and suppose that a point P in the fluid may be specified by cylindrical polar co-ordinates (r, θ, z) . Thus, all the quantities associated with the flow are independent of θ . The equation of continuity in cylindrical co-ordinates, becomes

$$\frac{\partial}{\partial r}(rq_r) + \frac{\partial}{\partial z}(rq_z) = 0$$

i.e.
$$\frac{\partial}{\partial r}(rq_r) = -\frac{\partial}{\partial z}(rq_z) \qquad (1)$$

This is the condition of exactness of the differential equation

$$rq_r dz - rq_z dr = 0 \qquad (2)$$

This means that (2) is an exact differential equation and let L.H.S. be an exact differential $d\varphi$(say)

Therefore,

$$rq_r dz - rq_z dr = d\varphi = \frac{\partial\varphi}{\partial r}\,dr + \frac{\partial\varphi}{\partial z}\,dz$$

which gives
$$\frac{\partial\varphi}{\partial r} = rq_z\ , \frac{\partial\varphi}{\partial z} = rq_r \qquad (3)$$

The function φ in (3) is called **Stoke's stream function**.

The equation of streamlines in the meridian plane $\theta = constant$ at a fixed time t is

$$\frac{dz}{q_z} = \frac{dr}{q_r}$$

$$\Rightarrow \qquad q_z\,\mathrm{dr} = q_r dz$$

Using (3), we get

$$-\frac{1}{r}\frac{\partial\varphi}{\partial r}\,dr = \frac{1}{r}\frac{\partial\varphi}{\partial z}\,dz$$

$\Rightarrow$ $$\frac{\partial\varphi}{\partial r}\,dr + \frac{\partial\varphi}{\partial z}\,dz = 0$$

$\Rightarrow$ $d\varphi = 0$

$\Rightarrow$ $\varphi = constant = C$

which represent the streamlines.

14.4 *PROPERTY OF STOKES FNUCTION*

In fluid dynamics and potential theory, the Stokes function refers to a fundamental solution to the Stokes equations, which describe the motion of a viscous fluid. Here's a key property of the Stokes function:

Green's Function property

The Stokes function serves as a Green's function for the Stokes equations in certain domains. Specifically, it satisfies the following integral equation:

$$\int_V G(x,y)\nabla^2 u(y)dy = u(x)$$

- $G(\mathbf{x},\mathbf{y})$ is the Stokes function, representing the velocity field generated by a point force or source located at y in a viscous fluid.
- $\nabla^2 u(y)$ is the Laplacian of the velocity field $u(y)$.
- $u(x)$ is the velocity field at the point x.

This property essentially states that the Stokes function (x,y) acts as an inverse operator to the Laplacian ∇^2. It provides a solution to the problem of finding the velocity field $u(x)$ in a viscous fluid due to a localized source or force at y.

Moreover, the Stokes function typically satisfies additional conditions such as boundary conditions specific to the problem domain, ensuring that it accurately represents the physical behavior of the fluid flow. It is a fundamental tool in the theoretical and computational study of viscous fluid flows, used extensively in problems ranging from fluid dynamics to biological fluid mechanics and microfluidics.

14.5 *IMAGE OF A SOURCE IN A SPHERE*

Suppose a source of strength m is situated at point A at a distance f(> a) from the centre of the sphere of radius a.

Let B be the inverse point of A w.r.t. the sphere, then OB = a^2/f

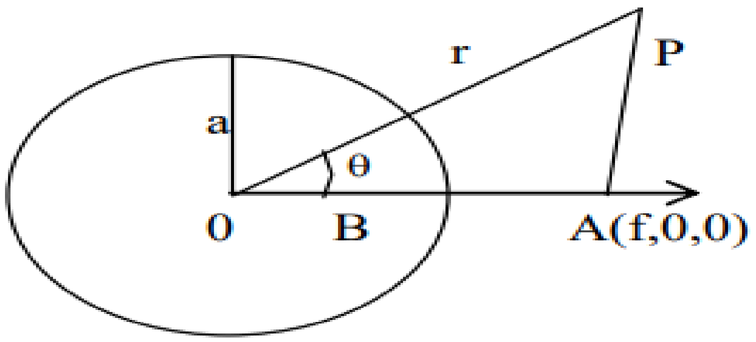


Figure 1

The velocity potential at P(r, θ, φ) in the fluid due to a simple source of strength m at A(f, 0, 0) is

$\emptyset(r, \theta) = \frac{m}{AP}$

From ΔOAP, $\cos\theta = \frac{(OP)^2+(OA)^2-(AP)^2}{2(OP)(OA)} = \frac{(r)^2+(f)^2-(AP)^2}{2(r)(f)}$

$\Rightarrow$ AP= $\sqrt{(r)^2 + (f)^2 - 2rfcos\theta}$

Thus, the velocity potential is

$$\emptyset(r, \theta) = m((r)^2 + (f)^2 - 2rfcos\theta)^{-1/2}$$

Introducing a solid sphere in the region r $\leq a$, where a < f, we obtain on using Weiss's sphere theorem, a perturbation potential

$$\frac{a}{r}\emptyset\left(\frac{a^2}{r},\theta\right)-\frac{1}{a}\int_0^{a^2/r}\emptyset(R,\theta)dR$$

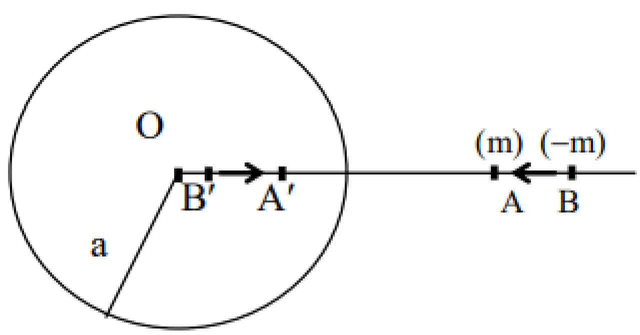
i.e. $$\frac{am}{r}\left[\frac{a^4}{r^2}+f^2-2\frac{a^2}{r}f\cos\theta\right]^{-1/2}-\frac{m}{a}\int_0^{a^2/r}[(R)^2+(f)^2-2Rf\cos\theta]^{-1/2}dR$$

i.e $$\frac{(ma/f)}{\sqrt{r^2-2r\left(\frac{a^2}{f}\right)\cos\theta+\left(\frac{a^2}{f}\right)^2}}-\frac{m}{a}\int_0^{a^2/r}\frac{dR}{\sqrt{[(R)^2+(f)^2-2Rf\cos\theta]}}$$

This shows that the image system of a point source of strength m placed at distance f(> a) from the centre of solid sphere consists of a source of strength $\frac{ma}{f}$ at the inverse point a^2/f in the sphere, together with a continuous line distribution of sinks of uniform strength $\frac{m}{a}$ per unit length extending from the centre to the inverse point.

14.6 *IMAGE OF A DOUBLET IN A SPHERE WHEN THE AXIS OF THE DOUBLET PASSES THROUGH THE CENTRE OF THE SPHERE*

Let us consider a doublet AB with its axis $\overline{BA}$ pointing towards the centre 0 of a sphere of radius a. Let OA = f, OB = f+δf. Let A',B' be the inverse points of A & B in the sphere so that



OA′ = a²/f, OB′ = a²/(f+δf).

Figure 2

At A, B we associate simple sources of strengths m and –m so that the strength of the double is $\mu = m\delta f$, where μ is to remain a finite non-zero constant as m→ ∞ and $\delta f \to 0$ simultaneously.

B'A'=OA'-OB'= $\frac{a^2}{f} - \frac{a^2}{f+\delta f} = \frac{a^2}{f}\left(\left(1+\frac{\delta f}{f}\right)^{-1}\right)$

$= \frac{a^2}{f} - \frac{a^2}{f} + \frac{a^2}{f}\frac{\delta f}{f}$ to the first order

$=\frac{a^2}{f^2}\delta f$ to the first order

Now, from the case of "Image of source in a sphere", the image of m at A consists of m at A consists of $\frac{ma}{f}$ at A' together with a continuous line distribution from O to A′ of sinks of strength $\frac{m}{a}$ per unit length and the image of –m at B consists of $-\frac{ma}{f+\delta f}$ at B' together with a continuous line distribution from O to B′ of sources of strength $\frac{m}{a}$ per unit length.

The line distribution of sinks and sources from 0 to B' cancel each other leaving behind a line distribution of sinks of strength $\frac{m}{a}$ per unit length from B' to A' i.e. sink of strength $\frac{m}{a}$ B'A'= $\frac{m}{a}\left(\frac{a^2}{f^2}\delta f\right) = \frac{a}{f^2}(m\delta f) = \frac{\mu a}{f^2}$, at B'. The source at B' is of strength

$$-\frac{ma}{f+\delta f} = \frac{-ma}{f}\left(\left(1+\frac{\delta f}{f}\right)^{-1}\right) = \ = \frac{-ma}{f}\left(1-\frac{\delta f}{f}\right) \ ,$$

to the first order terms

$$=\frac{-ma}{f} + \frac{ma}{f^2}\delta f = \frac{-ma}{f} + \frac{\mu a}{f^2}$$

Which is equivalent to a sink $\frac{ma}{f}$ at B' and a source $\frac{\mu a}{f^2}$ at B'.

As there is already a sink $\frac{\mu a}{f^2}$ at B', therefore source and sink at B′ neutralize.

Finally, we are left with source $\frac{ma}{f}$ at A' and a sink.f ma and a sink. $\frac{ma}{f}$ at B′. Thus, to the first order, we obtain a doublet at A' of strength

$$\frac{ma}{f}(B'A') = \frac{ma}{f}\frac{a^2}{f^2}\delta f$$

$$= \frac{ma^3}{f^3}\delta f \ = \frac{\mu a^3}{f^3} \ .$$

Hence in the limiting case as $\delta f \to 0$ *and* m→ ∞ , we obtain a double at A of strength μ with its axis towards O, together with a doublet at the inverse point A' of strength $\frac{\mu a^3}{f^3}$ with its axis away from O.

14.7 *EXAMPLES*

EXAMPLE1: In a flow field, the velocity components were evaluated as

u = a(x^2-y^2), v = -2axy, w = 0.

Check whether you can form a stream function for this flow field. If so, what is the stream function?

SOLUTION:

Given: u = a(x^2-y^2), v = - 2axy, w = 0.

As w = 0, the flow is 2-dimensional, we need to check whether the flow is incompressible.

(Note: For incompressible fluid $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$)

$$\frac{\partial u}{\partial x} = 2ax, \quad \frac{\partial v}{\partial y} = -2ax$$

$$so, \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

That is fluid is incompressible and the continuity equation $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ is satisfied.

Therefore, we can define the scalar stream function ψ for the given problem.

As u= $\frac{\partial \varphi}{\partial y}$ = a(x^2-y^2)

$\Rightarrow \varphi = \int u\, dy + f(x)$

$\Rightarrow \varphi = ax^2$y- a $\frac{y^3}{3}$ + f(x) …..(1)

$\Rightarrow \frac{\partial \varphi}{\partial x}$ = 2axy+ f'(x) …..(2)

You know v=- $\frac{\partial \varphi}{\partial x}$ = -2axy

$\Rightarrow \frac{\partial \varphi}{\partial x}$ = 2axy

Here in (2) f'(x)=0

Or, f(x)= Constant C

$\Rightarrow \varphi = ax^2$y- a $\frac{y^3}{3}$ + C

EXAMPLE 2: A stream function is given by $\psi = 3x^2 - y^3$.

Determine the magnitude of velocity components at the point (3,1).

SOLUTION: The x and y components of velocity are given by

x- component: u= $\frac{\partial \varphi}{\partial y} = \frac{\partial(3x^2-y^3)}{\partial y}$ = - $3y^2$ …(1)

y – component: v= - $\frac{\partial \varphi}{\partial x}$ =- $\frac{\partial(3x^2-y^3)}{\partial x}$ = - 6x …(2)

At the point (3,1)

U=-3 and v= -18

and the total velocity is the vector sum of the two components.

V= -3 i- 18 j

Note that $\partial u/\partial x = 0$ and $\partial v/\partial y$=0, so that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Therefore the given stream function satisfies the continuity equation.

The equation for vorticity,

$\varepsilon = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ …(3)

may also be expressed in terms of by substituting Eqs. (1) and (2) into Equ. (3)

$\varepsilon = \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial y^2}$

However, for irrotational flows, and the classic Laplace equation,

$$\frac{\partial^2\varphi}{\partial x^2} - \frac{\partial^2\varphi}{\partial y^2} = \nabla^2\varphi = 0$$

results. This means that the stream functions of all irrotational flows must satisfy the Laplace equation and that such flows may be identified in this manner; conversely, flows whose does not satisfy the Laplace equation are rotational ones. Since both rotational and irrotational flow fields are physically possible, the satisfaction of the Laplace equation is no criterion of the physical existence of a flow field.

EXAMPLE 3: A flow field is described by the equation $\psi = y\text{-}x^2$. Sketch the streamlines $\psi = 0$, $\psi=1$, and $\psi = 2$. Derive an expression for the velocity V at any point in the flow field. Calculate the vorticity.

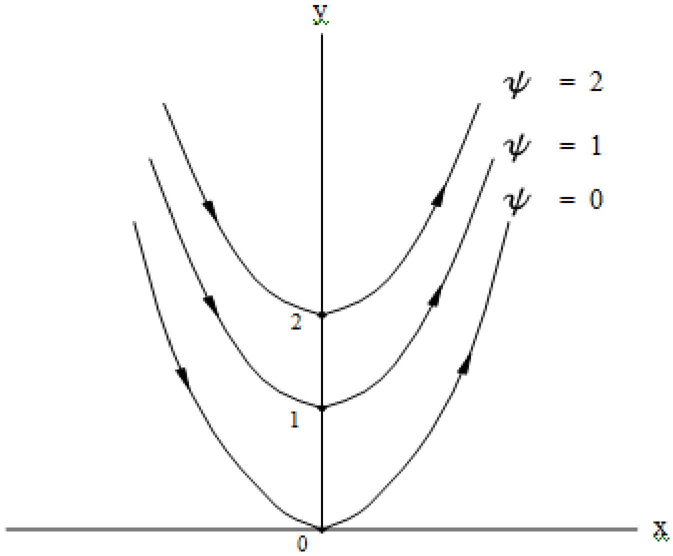


Fig. 14.7.3.1

SOLUTION: From the equation for ψ, the flow field is a family of parabolas symmetrical about the y-axis with the streamline $\varphi = 0$ passing through the origin of coordinates.

$$U = \frac{\partial\varphi}{\partial y} = \frac{\partial(y-x^2)}{\partial y} = 1 \quad V = -\frac{\partial\varphi}{\partial x} = -\frac{\partial(y-x^2)}{\partial x} = 2x$$

Which allows the directional arrows to be placed on streamlines as shown.

The magnitude V of the velocity may be calculated from v=$\sqrt{u^2+v^2}$ =$\sqrt{1+4x^2}$

and the vorticity by Equation (3)

$\varepsilon = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial(2x)}{\partial x} - \frac{\partial(1)}{\partial y} = 2\sec^{-1}\emptyset$ (Counter clockwise)

Since $\zeta \neq 0$, this flow field is seen to be rotational one.

EXAMPLE 4: A stream function in a two-dimensional flow is φ = 2xy. Show thatthe flow is irrotational (potential) and determine the corresponding velocity potential function.

SOLUTION: The given stream function satisfies the condition of irrotationality, that is,

$$w = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial y^2}\right)$$

$$= \frac{1}{2}\left(\frac{\partial^2(2xy)}{\partial x^2} - \frac{\partial^2(2xy)}{\partial y^2}\right) = 0$$

which shows that the flow is irrotational. Therefore, a velocity potential function will exist for this flow.
By using Equation.

$$\frac{\partial \varphi}{\partial y} = \frac{\partial \emptyset}{\partial x} = \frac{\partial(2xy)}{\partial y} = 2x$$

Therefore,

$$\emptyset = \int 2x \partial x = x^2 + f_1(y) \qquad \text{.....(a)}$$

From Equation

$$\frac{\partial \emptyset}{\partial y} = -\frac{\partial \varphi}{\partial x} = -\frac{\partial(2xy)}{\partial x} = -2y$$

From this equation,

$$\emptyset = \int 2y \partial y = y^2 + f_2(x) \qquad \text{.....(b)}$$

The velocity potential function,

$$\emptyset = x^2 - y^2 + C$$

satisfies both functions in Equations a and b.

14.8 *SUMMARY*

(i) Stoke's Stream Function Stokes function.

(ii) Image of source relative to sphere

(iii) Image of doublet relative to sphere

14.9 GLOSSARY

(i). Fluid

(ii). Two-Dimensional Flow

(iii). Cylindrical Polar Coordinates

(iv). Viscous fluid: is a real fluid that flows with some resistance in the opposite direction of its flow.

14.10 *REFERENCES AND SUGGESTED READING*

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14.11 TERMINAL QUESTIONS

Question 1. Define Property of Stokes function.

Question 2. What is Stoke's Stream Function.

Question 3. A stream function is given by $\psi = 3x^5 - y^3$.

Determine the magnitude of velocity components at the point (3,1).

Question 4. What is Green's Function property in strain.

Question 5. Write the difference between Image of source relative to sphere and Image of doublet relative to sphere.

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